DERIVATION OF NON-NEWTONIAN MAGNETIC FLUID LUBRICATED ROUGH CENTROSYMMETRIC SQUEEZE FILM REYNOLDS EQUATION AND ITS APPLICATION

J. R. Lin*

Nanya Institute of Technology
Taoyuan, Taiwan

L. M. Chu

National Taitung University
Taitung, Taiwan

H. L. Chiang **Y. K. Chiu**

Nanya Institute of Technology
Taoyuan, Taiwan

ABSTRACT

Based upon the Shliomis ferromagnetic fluid model and the Stokes microcontinuum theory incorporating with the Christensen stochastic model, a modified Reynolds equation of centrosymmetric squeeze films has been derived in this paper. The Reynolds equation includes the combined effects of non-Newtonian rheology, magnetic fluids with applied magnetic fields, rotational inertia forces, and surface roughness. To guide the use of the derived equation, the squeeze film of rotational rough-surface circular disks lubricated with non-Newtonian magnetic fluids is illustrated. According to the results obtained, the effects of rotation inertia decrease the load capacity and the squeeze film time of smooth circular disks. By the use of non-Newtonian magnetic fluids with applied magnetic fields, the rotational circular disks predict better squeeze film performances. When the influences of circumferential roughness patterns are considered, the non-Newtonian magnetic-fluid lubricated rotational rough disks with applied magnetic fields provide further higher values of the load capacity and the squeeze film time as compared to those of the smooth case.

Keywords: Magnetic fluids, Non-Newtonian rheology, Rotational inertia, Surface roughness.

1. INTRODUCTION

From the contributions of Shliomis [1, 2] and Rosensweig [3], magnetic fluids (also called ferrofluids) are colloidal solutions of nano ferromagnetic particles coated with a surfactant and dispersed in a liquid carrier. In the presence of external magnetic fields, the ferromagnetic nano-particles can experience a body force oriented along the field line direction. Because of their peculiar characteristics, magnetic fluids find a variety of applications in engineering sciences, such as the drug targeting [4], the emission patterns [5], the support grinding [6], the spin coating [7], the fluid film bearings [8-10]. Due to the development of modern engineering, a Newtonian fluid blended with small amounts of additives has gained great attention. Experimental evidences of Oliver [11] have shown that the polymer additives dissolved in the lubricant provide an enhancement in the load capacity for bearing systems. To simulate the flow behavior of these kinds of fluids with substructures, Stokes [12] has generated a microcontinuum theory of non-Newtonian couple stress fluid mode. Applying the Stokes microcontinuum theory, many contributions have been presented, such as the human joint lubrication [13], the suspension flow of red blood cells [14], and the peristaltic pumping flow [15]. In addition, Lin *et al*. [16] investigate the squeeze film performances between smooth-surface disks with a non-Newtonian magnetic fluid. However, bearing surfaces may be rough in practice owing to the manufacturing process.

* Corresponding author (jrlin@nanya.edu.tw)

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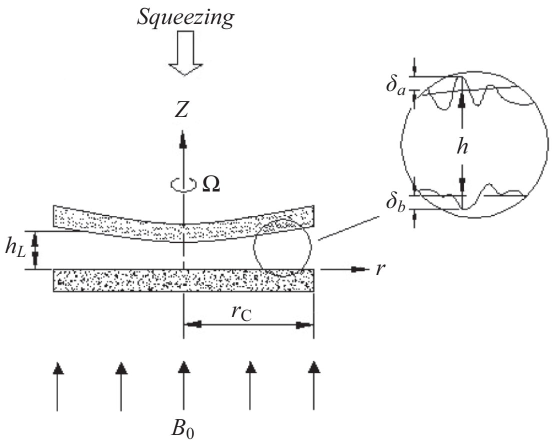

Fig. 1 Squeeze film geometry of rotational rough curved circular disks lubricated with a non-Newtonian ferrofluid.

Therefore, the influences of surface roughness should be considered in the study of squeeze films. Introducing the concept of a stochastic model, Christensen [17] has generated an averaging film model for striated roughness of surfaces. Applying the Christensen stochastic model, a number of contributors investigated the surface roughness effects on the squeeze film bearings [18-20]. Recently, Lin *et al.* [21] investigates the squeeze film characteristics between rough circular disks lubricated by magnetic fluids with non-Newtonian rheology. However, this study neglects the effects of rotational inertia forces. For engineering practices, the upper disk of squeeze films is often rotational. Therefore, a further study including the of rotational inertia forces is motivated.

Based upon the Shliomis ferromagnetic fluid model [1, 2] together with the Stokes microcontinuum theory [12] and the Christensen stochastic model [17], the present study is to derive a modified Reynolds equation of centrosymmetric curved squeeze films for engineering applications. To guide the use of the derived Reynolds equation, a circular squeeze film is illustrated. Comparing with the non-rotational smooth disks with a Newtonian non-magnetic fluid, the squeeze film performances are presented through the variation of the volume concentration of particles, the magnetic Langevin parameter, the non-Newtonian couple stress parameter, the surface roughness parameter and the rotational parameter.

2. DERIVATION OF LUBRICATION EQUATION

Figure 1 shows the squeeze film geometry of rotational rough centrosymmetric curved disks lubricated with an incompressible non-Newtonian magnetic fluid in the presence of a constant magnetic field $\vec{B} = (0, 0, B_0)$. The upper curved disk with a rotating angular speed Ω approaches the lower one with a squeezing velocity $w_{sq} = -\partial h_L/\partial t$, where t denotes the time, and the local film thickness h_L is made up of two parts,

$$h_L = h(t) + \delta(r, \theta, \zeta) \tag{1}$$

In this equation, h denotes the nominal smooth part of the lubricant film, and $\delta = \delta_a + \delta_b$ represents the random part owing to the surface asperities, and ζ is a random variable describing the roughness arrangement.

For the present study, it is assumed that the film thickness is much smaller than radius curvature and the thin-film lubrication theory is applicable. Based upon the Shliomis ferromagnetic fluid model [1, 2] together with the Stokes microcontinuum theory [12], the equations of momentum and continuity in axially centrosymmetric coordinates (r, θ, z) can be reduced to the following form.

$$\eta_c \frac{\partial^4 u}{\partial z^4} - \eta \frac{\partial^2 u}{\partial z^2} - \frac{3V(L - \tanh L)}{2(L + \tanh L)} \eta \frac{\partial^2 u}{\partial z^2} = \frac{\rho v^2}{r} - \frac{\partial p_L}{\partial r} \tag{2}$$

$$\eta_c \frac{\partial^4 v}{\partial z^4} - \eta \frac{\partial^2 v}{\partial z^2} = 0 \tag{3}$$

$$\frac{\partial p}{\partial z} = 0 \tag{4}$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

In these equations u, v and w are the velocity components in the r–, θ– and z–directions respectively, ρ is the mass density, p_L is the local film pressure, η is the viscosity of the suspension, η_c is a material constant for the non-Newtonian couple stress fluids, V is the volume concentration of nano ferromagnetic particles, and L denotes the magnetic Langevin parameter,

$$L = \frac{m\mu_p B_0}{k_B \Gamma} \tag{6}$$

where m is the magnetic moment of a particle, μ_p is the free space permeability, k_B is the Boltzmann constant, and Γ is the absolute temperature. According to the research of Batchelor [22], the effective viscosity of the suspension is approximately found to be: $\eta = \eta_m\alpha$, where η_m represents the viscosity of the main liquid, and

$$\alpha = 1 + 2.5V + 6.2V^2 \tag{7}$$

The non-slip conditions and the non-couple stress conditions for velocity components are:

$$\text{at} \quad z = 0: \quad u = 0, \quad v = 0, \quad w = 0, \quad \partial^2 u/\partial z^2 = 0, \quad \partial^2 v/\partial z^2 = 0 \tag{8}$$

$$\text{at} \quad z = h_L: \quad u = 0, \quad v = r\Omega, \quad w = -w_{sq} = \partial h_L/\partial t, \quad \partial^2 u/\partial z^2 = 0, \quad \partial^2 v/\partial z^2 = 0 \tag{9}$$

Solving Eq. (3) and applying the boundary conditions of v, one can obtain: $v = r\Omega z/h_L$. Solving Eq. (2) and applying the expression of v and the boundary conditions of u, one can obtain

$$u = u_A + u_B \tag{10}$$

where

$$u_A = \frac{1}{2\eta_m \alpha(1+\beta)} \frac{\partial p_L}{\partial r} \left\{ z^2 - h_L z + \frac{2l_c^2}{\alpha(1+\beta)} \left\{ 1 - \frac{\cosh[(2z-h_L)\sqrt{\alpha(1+\beta)}/2l_c]}{\cosh[h_L\sqrt{\alpha(1+\beta)}/2l_c]} \right\} \right\} \tag{11}$$

$$u_B = \frac{-\rho\Omega^2 r}{12\eta_0 \alpha(1+\beta) h_L^2} \left\{ \begin{array}{l} z^4 + \dfrac{12l_c^2}{\alpha(1+\beta)} \left\{ z^2 - \dfrac{\sinh[z\sqrt{\alpha(1+\beta)}/l_c]}{\sinh[h_L\sqrt{\alpha(1+\beta)}/l_c]} h_L^2 \right\} \\ -h_L^3 z + \dfrac{24l_c^4}{\alpha^2(1+\beta)^2} \left\{ 1 - \dfrac{\cosh[(2z-h_L)\sqrt{\alpha(1+\beta)}/2l_c]}{\cosh[h_L\sqrt{\alpha(1+\beta)}/2l_c]} \right\} \end{array} \right\} \tag{12}$$

$$l_c = \sqrt{\eta_c / \eta_m} \tag{13}$$

$$\beta = \frac{3V(L - \tanh L)}{2(L + \tanh L)} \tag{14}$$

Integrating the continuity Eq. (5) across the film yields

$$\int_0^{h_L} \frac{\partial}{\partial r}(ru)dz = -r\int_0^{h_L} \frac{\partial w}{\partial z} dz \tag{15}$$

Performing the integration and applying the velocity boundary conditions, one can obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{ 10 r g_A(\alpha,\beta,l_c,h_L)\frac{\partial p_L}{\partial r} - 3\rho\Omega^2 r^2 g_B(\alpha,\beta,l_c,h_L) \right\} = 120\alpha(1+\beta)\eta_m \frac{\partial h_L}{\partial t} \tag{16}$$

where

$$g_A(\alpha,\beta,l_c,h_L) = h_L^3 - \frac{12l_c^2}{\alpha(1+\beta)} \left\{ h_L - \frac{2l_c}{\sqrt{\alpha(1+\beta)}} \tanh\left[\frac{\sqrt{\alpha(1+\beta)}h_L}{2l_c} \right] \right\} \tag{17}$$

$$g_B(\alpha,\beta,l_c,h_L) = \left\{ \begin{array}{l} h_L^3 - \dfrac{40l_c^2 h_L}{3\alpha(1+\beta)} + \dfrac{40l_c^3}{\alpha^{1.5}(1+\beta)^{1.5}} \tanh\left[\dfrac{\sqrt{\alpha(1+\beta)}h_L}{2l_c} \right] \\ -\dfrac{80l_c^4}{\alpha^2(1+\beta)^2 h_L} + \dfrac{160l_c^5}{\alpha^{2.5}(1+\beta)^{2.5} h_L^2} \tanh\left[\dfrac{\sqrt{\alpha(1+\beta)}h_L}{2l_c} \right] \end{array} \right\} \tag{18}$$

Apply the Christensen stochastic model [17] for rough surfaces and taking the expected values of both sides of Eq. (16), one can derive a modified Reynolds equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{ 10rE\left[g_A(\alpha,\beta,l_c,h_L)\right]\frac{\partial E\left[p_L\right]}{\partial r} - 3\rho\Omega^2 r^2 E\left[g_B(\alpha,\beta,l_c,h_L)\right] \right\} = 120\alpha(1+\tau)\eta_m \frac{\partial E\left[h_L\right]}{\partial t} \tag{19}$$

where the expectancy operator E is defined by an integral.

$$E[\bullet] = \int_{-\infty}^{\infty} [\bullet] f(\delta) d\delta \tag{20}$$

In this equation, δ denotes the stochastic variable, and $f(\delta)$ describes the probability density distribution for the stochastic variable. Since most engineering rough surfaces are Gaussian in nature, for simplicity one could choose a polynomial density function as Christensen [17] to approach the Gaussian distribution.

$$f(\delta) = \begin{cases} \dfrac{35(c^2-\delta^2)}{32c^7}, & if\ -c \le \delta \le c \\ 0, & elsewhere \end{cases} \tag{21}$$

where c describes the half total range of random film-thickness variable.

3. APPLICATION TO CIRCULAR SQUEEZE FILM

As an example of the derived Reynolds Eq. (19), we consider the squeeze film between rotational rough-surface circular disks lubricated with a non-Newtonian magnetic fluid in the presence of a transverse magnetic field. In addition, the effects of one-dimensional circumferential roughness patterns are investigated. In this situation, the circular disks possess the form of narrow ridges and valleys running in the θ-direction.

$$h_L = h(t) + \delta(r, \chi) \tag{22}$$

where χ denotes the random variable. Then, the stochastic non-Newtonian magnetic-fluid Reynolds Eq. (19) for the rotational disks can be expressed as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{10r}{E\left[g_A^{-1}(\alpha,\beta,l_c,h_L)\right]}\frac{\partial \bar{p}}{\partial r} - \frac{3\rho\Omega^2 r^2}{E\left[g_B^{-1}(\alpha,\beta,l_c,h_L)\right]}\right\} = 120\alpha(1+\tau)\eta_m \frac{\partial E\left[h_L\right]}{\partial t} \tag{23}$$

where $\bar{p} = E[p_L]$ is the mean film pressure. Applying the definition of the expectancy operator, one can achieve

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{10rG_A\frac{\partial \bar{p}}{\partial r} - 3\rho\Omega^2 r^2 G_B\right\} = 12\alpha(1+\tau)\eta_m \frac{dh}{dt} \tag{24}$$

where

$$G_A(\alpha,\beta,l_c,h,c) = \left[\frac{35}{32c^7}\int_{\delta=-c}^{+c} g_A^{-1}(c^2-\delta^2)^3\, d\delta\right]^{-1} \tag{25}$$

$$G_B(\alpha,\beta,l_c,h,c) = \left[\frac{35}{32c^7}\int_{\delta=-c}^{+c} g_B^{-1}(c^2-\delta^2)^3\, d\delta\right]^{-1} \tag{26}$$

In order to conveniently analyze the problem, the non-dimensional quantities are introduced as follows.

$$R = \frac{r}{r_c},\quad H = \frac{h}{h_0},\quad P = \frac{\bar{p}h_0^3}{\eta_m r_c^2(-dh/dt)},\quad \Delta = \frac{\delta}{h_0},\quad G_1 = \frac{G_A}{h_0^3},\quad G_2 = G_B h_0^3 \tag{27}$$

$$N = \frac{l_c}{h_0},\quad \gamma = \frac{\rho h_0^3\Omega^2}{\eta_m(-dh/dt)},\quad \Lambda = \frac{c}{h_0} \tag{28}$$

where h_0 denotes the initial smooth film thickness, γ is the rotational parameter, and Λ is the surface roughness parameter. Then the modified Reynolds equation is expressed in a non-dimensional form.

$$\frac{1}{R}\frac{\partial}{\partial R}\left\{10RG_1\frac{\partial P}{\partial R} - 3\gamma G_2 R^2\right\} = -120\alpha(1+\tau) \tag{29}$$

where $G_1 = G_1(\alpha, \beta, N, H, \Lambda)$ and $G_2 = G_2(\alpha, \beta, N, H, \Lambda)$

$$G_1 = \left[\frac{35}{32\Lambda^7}\int_{-\Lambda}^{\Lambda}\frac{(\Lambda^2-\Delta^2)^3}{G_{11}+G_{12}}\,d\Delta\right]^{-1} \tag{30}$$

$$G_2 = \left[\frac{35}{32\Lambda^7}\int_{-\Lambda}^{\Lambda}\frac{(\Lambda^2-\Delta^2)^3}{G_{21}+G_{22}+G_{23}}\,d\Delta\right]^{-1} \tag{31}$$

$$G_{11} = (H+\Delta)^3 - \frac{12N^2}{\alpha(1+\beta)}(H+\Delta) \tag{32}$$

$$G_{12} = \frac{24N^3}{\alpha^{3/2}(1+\beta)^{3/2}}\tanh\left[\frac{(H+\Delta)\sqrt{\alpha(1+\tau)}}{2N}\right] \tag{33}$$

$$G_{21} = (H+\Delta)^3 - \frac{40N^2}{3\alpha(1+\beta)}(H+\Delta) \tag{34}$$

$$G_{22} = \frac{40N^3}{\alpha^{3/2}(1+\beta)^{3/2}} \tanh\left[\frac{(H+\Delta)\sqrt{\alpha(1+\beta)}}{2N}\right] - \frac{80N^4}{\alpha^2(1+\beta)^2}\frac{1}{(H+\Delta)} \tag{35}$$

$$G_{23} = \frac{160N^5}{\alpha^{5/2}(1+\beta)^{5/2}}\frac{1}{(H+\Delta)^2} \tanh\left[\frac{(H+\Delta)\sqrt{\alpha(1+\beta)}}{2N}\right] \tag{36}$$

The non-dimensional mean film pressure can be obtained by integrating the Reynolds equation subject to the boundary conditions: $dP/dR = 0$ at $R = 0$, and $P = 0$ at $R = 1$.

$$P = \frac{60\alpha(1+\beta) - 3\gamma G_2}{20G_1}(1-R^2) \tag{37}$$

The mean load capacity is evaluated by integrating the mean film pressure acting on the upper disk,

$$\bar{w} = \int_0^{r_c} 2\pi r\bar{p}dr \tag{38}$$

Using a non-dimensional form, one can obtain the non-dimensional mean load capacity,

$$W = \frac{\bar{w}h_0^{\,3}}{\eta_m r_c^{\,4}(-dh/dt)} = \frac{60\pi\alpha(1+\tau) - 3\pi\gamma G_2}{40G_1} \tag{39}$$

Introduce a non-dimensional form for the squeeze film time, $T = \bar{w}h_0^2 t/\eta_m r_c^4$. Substituting into the equation of the non-dimensional mean load capacity, one can achieve

$$\frac{dH}{dT} = -\frac{40G_1}{60\pi\alpha(1+\beta) - 3\pi\gamma G_2} \tag{40}$$

The initial condition for the non-dimensional film height is: $H(T = 0) = 1$. Separating the variables and integrating the equation, solution for the non-dimensional squeeze film time can be obtained,

$$T = \frac{3\pi\alpha(1+\beta)}{2}\int_H^1 \frac{1}{G_1}dH - \frac{3\pi\gamma}{40}\int_H^1 \frac{G_2}{G_1}dH \tag{41}$$

The double integrals appearing in the above equation can be calculated by the method of Gaussian quadrature.

4. RESULTS AND DISCUSSION

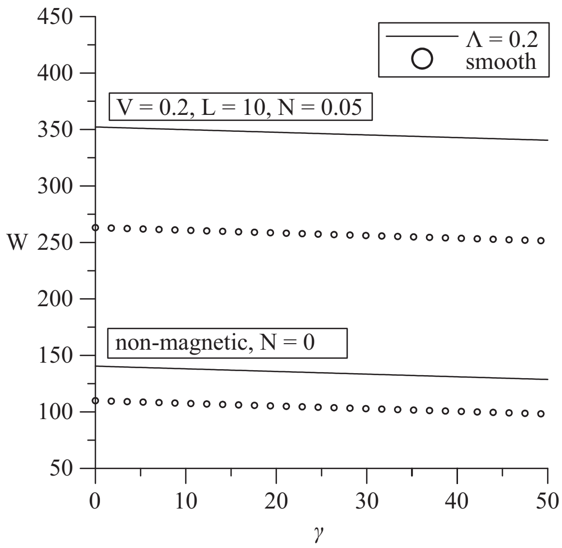

Fig. 2 Mean load capacity W versus rotational parameter γ for different V, L, N and Λ.

Figure 2 describes the non-dimensional mean load capacity W versus the rotational parameter γ for circular disks operating at $H = 0.35$ for different values of the volume concentration V, the magnetic Langevin parameter L, he non-Newtonian couple stress parameter N and the surface roughness parameter Λ. It is shown that the mean load capacity decreases with increasing values of the rotational parameter. Comparing with the non-magnetic Newtonian case, the non-Newtonian magnetic-fluid lubricated disks with applied magnetic fluid ($V = 0.2$, $L = 10$, $N = 0.05$) are observed to result in a higher load capacity. It is also observed that the circumferential roughness effects ($\Lambda = 0.2$) predict further higher values of the mean load capacity.

Figure 3 describes the non-dimensional mean squeeze film time T versus the volume concentration of particles V operating at $H = 0.35$ for different values of L, N, γ and Λ. The mean squeeze film time is seen to increase with increasing values of the volume concentration parameter. Comparing with the case of Newtonian non-rotational smooth disks without magnetic fields ($L = 0$, $N = 0$, $\gamma = 0$, $\Lambda = 0$), the application of an external magnetic field ($L = 10$, $N = 0$, $\gamma = 0$, $\Lambda = 0$) yields a longer value of T. In addition, the non-Newtonian influences of couple stresses ($L = 10$, $N = 0.05$, $\gamma = 0$, $\Lambda = 0$) are observed to

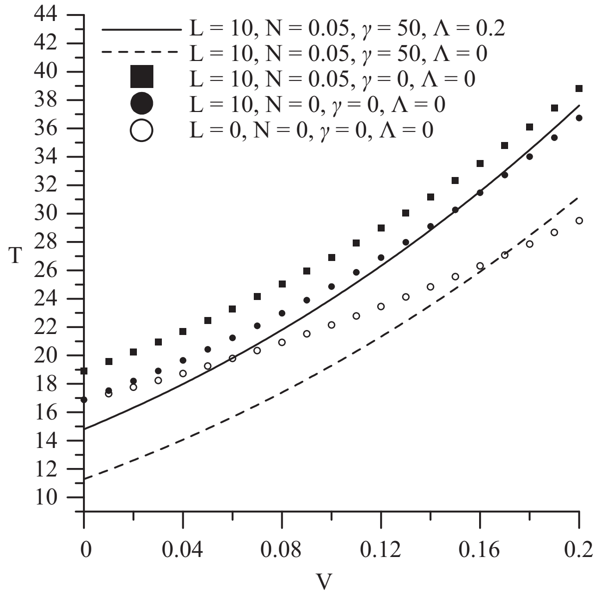

Fig. 3 Mean squeeze film time T versus concentration parameter V for different L, N, γ and Λ.

provide more increments of the mean squeeze film time. When the effects of rotational inertia are considered (L = 10, N = 0.05, γ = 50, Λ = 0), shorter squeeze film time are obtained. However, the roughness patterns of circumferential structures (L = 10, N = 0.05, γ = 50, Λ = 0.2) provide an increment in the mean squeeze film time as compared to the non-Newtonian magnetic fluid lubricated rotational smooth disks (L = 10, N = 0.05, γ = 50, Λ = 0).

5. CONCLUSIONS

Based upon the Shliomis ferromagnetic fluid model and the Stokes microcontinuum theory together with the Christensen stochastic model, a modified Reynolds equation of centrosymmetric squeeze films (including the combined effects of non-Newtonian rheology, magnetic fluids with applied magnetic fields, rotational inertia forces, surface roughness) has been derived for engineering applications. To guide the use of the derived Reynolds equation, the squeeze film of rotational rough circular disks with non-Newtonian magnetic fluids is illustrated. It is shown that the effects of rotation inertia decrease the load capacity and the squeeze film time of smooth circular disks. By the use of non-Newtonian magnetic fluids with applied magnetic fields, the rotational circular disks predict a higher load capacity and a longer squeeze film time. When the surface roughness of circumferential patterns are considered, the non-Newtonian magnetic-fluid lubricated rotational rough disks with applied magnetic fields provide further higher values of the load capacity and the squeeze film time than those of the smooth case.

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