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A lower bound for the dispersion on the torus

Mario Ullrich

Johannes Kepler University, Altenberger Str. 69, 4040 Linz, Austria

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Abstract

We consider the volume of the largest axis-parallel box in the d-dimensional torus that contains no point of a given point set $\mathcal{P}_n$ with n elements. We prove that, for all natural numbers d, n and every point set $\mathcal{P}_n$, this volume is bounded from below by $\min\{1, d/n\}$. This implies the same lower bound for the discrepancy on the torus.

Keywords: Dispersion; Discrepancy; Torus

1. Introduction

The study of uniform distribution properties of n-element point sets $\mathcal{P}_n$ in the d-dimensional unit cube has attracted a lot of attention in past decades, in particular because of its strong connection to worst case errors of numerical integration using cubature rules, see e.g. [5,13,16]. There is a vast body of articles and books considering the problem of bounding the discrepancy of point sets. That is, given a probability space (X, μ) and a set $\mathcal{B}$ of measurable subsets of X, which we call *ranges*, we want to find the maximal difference between the measure of a set $B \in \mathcal{B}$ and the empirical measure induced by the finite set $\mathcal{P}_n$, i.e.

$$D(\mathcal{P}_n, \mathcal{B}) := \sup_{B \in \mathcal{B}} \left| \frac{\#(\mathcal{P}_n \cap B)}{n} - \mu(B) \right|,$$

where $\mathcal{P}_n \subset X$, $n \in \mathbb{N}$, with $\#\mathcal{P}_n = n$. In what follows we only consider $X = [0, 1]^d$, $d \geq 1$, and the Lebesgue measure μ; we write $|B| := \mu(B)$. The number $D(\mathcal{P}_n, \mathcal{B})$ is called the *discrepancy* of the point set $\mathcal{P}_n$ with respect to the ranges $\mathcal{B}$. See e.g. the monographs/surveys [4–6,13,14,16] for the state of the art, open problems and further literature on this topic.

Here, we are interested in lower bounds for this quantity that hold for every point set $\mathcal{P}_n$. In fact, we are going to bound the apparently smaller quantity

$$\operatorname{disp}(\mathcal{P}_n, \mathcal{B}) := \sup_{\substack{B \in \mathcal{B}: \\ \mathcal{P}_n \cap B = \varnothing}} |B|,$$

E-mail address: mario.ullrich@jku.at.

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which we call the *dispersion* of the point set $\mathcal{P}_n$ with respect to the ranges $\mathcal{B}$. Clearly, this is a lower bound for the discrepancy.

The notion of the dispersion was introduced by Hlawka [9] as the radius of the largest empty ball (for a given metric). In this setting there are some applications including the approximation of extreme values (Niederreiter [12]) or stochastic optimization (Yakowitz et al. [19]). The present definition was introduced by Rote and Tichy [17] together with a treatment of its value for some specific point sets and ranges. Only recently an application to the approximation of high-dimensional rank one tensors was discussed in Bachmayr et al. [3] and Novak and Rudolf [15], where the ranges are all axis-parallel boxes in $[0,1]^d$. A polynomial-time algorithm for finding the largest empty axis-parallel box in dimension 2 was considered by Naamad, Lee and Hsu [11].

Our main interest is the complexity of the problem of finding point sets with small dispersion/discrepancy; especially the dependence on the dimension. That is, given some $\varepsilon > 0$ and $d \in \mathbb{N}$, we want to know how many points are necessary to achieve $\operatorname{disp}(\mathcal{P}_n, \mathcal{B}) \le \varepsilon$ or $D(\mathcal{P}_n, \mathcal{B}) \le \varepsilon$ for some $\mathcal{P}_n \subset [0,1]^d$ and $\mathcal{B} \subset 2^{[0,1]^d}$. For this we define the inverse functions

$$N_0(\varepsilon, \mathcal{B}) := \min\{n\colon \operatorname{disp}(\mathcal{P}, \mathcal{B}) \le \varepsilon \text{ for some } \mathcal{P} \text{ with } \#\mathcal{P} = n\}$$

and

$$N(\varepsilon, \mathcal{B}) := \min\{n\colon D(\mathcal{P}, \mathcal{B}) \le \varepsilon \text{ for some } \mathcal{P} \text{ with } \#\mathcal{P} = n\}.$$

We have $N_0(\varepsilon, \mathcal{B}) \le N(\varepsilon, \mathcal{B})$ for every $\varepsilon, \mathcal{B}$.

For example, if $\mathcal{B} = \mathcal{B}^d_{\mathrm{ex}}$ is the set of all axis-parallel boxes contained in $[0,1]^d$, then it is easily seen that for every point set there exists an empty box with volume larger than $1/(n+1)$; simply split the cube in $n+1$ equal parts, one of which must be empty. Moreover, it is known that with respect to the dependence on n this estimate is asymptotically optimal, i.e. there exists a sequence of point sets $(\mathcal{P}_n)_{n\in\mathbb{N}}$ such that $\operatorname{disp}(\mathcal{P}_n, \mathcal{B}^d_{\mathrm{ex}}) \le C_d/n$ for some $C_d < \infty$, see e.g. [17].[1]

However, if one considers increasing values of the dimension the situation is less clear: The best bounds to date are

$$\frac{\log_2 d}{4(n + \log_2 d)} \le \inf_{\mathcal{P}:\#\mathcal{P}=n} \operatorname{disp}(\mathcal{P}, \mathcal{B}^d_{\mathrm{ex}}) \le \frac{C^d}{n}$$

for some constant $C < \infty$, see Aistleitner et al. [2] for the lower bound and Larcher [10] for the upper bound. For a proof of a super-exponential upper bound see also Rote and Tichy [17, Prop. 3.1]. This can be rewritten as

$$(1/4 - \varepsilon)\frac{\log_2 d}{\varepsilon} \le N_0(\varepsilon, \mathcal{B}^d_{\mathrm{ex}}) \le \frac{C^d}{\varepsilon}.$$

Clearly, there is a huge difference in the behavior in d for the upper and the lower bound.

If we consider the discrepancy instead, then even the order in ε^{-1} differs in the upper and the lower bounds, i.e. for small enough $\varepsilon \le \varepsilon_0$ and all $d \in \mathbb{N}$ we have

$$c\, d\, \varepsilon^{-1} \le N(\varepsilon, \mathcal{B}^d_{\mathrm{ex}}) \le C\, d\, \varepsilon^{-2}$$

with some constants $0 < c, C < \infty$.[2] The lower bound is due to Hinrichs [8] and the upper bound was proven by Heinrich et al. [7]. To narrow the gap in the ε-behavior while keeping a polynomial behavior in d is a long-standing open problem, see also Novak and Woźniakowski [16] for more results/problems in this area.

Nevertheless, for fixed, small $\varepsilon > 0$ the d-dependence of $N(\varepsilon, \mathcal{B}^d_{\mathrm{ex}})$ is known to be linear. This motivates us to study the same problem for the dispersion. Unfortunately, we were not able to solve this problem for the ranges $\mathcal{B}^d_{\mathrm{ex}}$. Instead, we consider the "periodic" version of this problem.

More precisely, we regard the unit cube as the torus and consider the *periodic ranges* $\mathcal{B}^d_{\mathrm{per}}$ that are defined by

$$\mathcal{B}^d_{\mathrm{per}} := \left\{B_1(x,y)\colon x, y \in [0,1]^d\right\}, \tag{1}$$

[1] Note that for the discrepancy such an inequality cannot hold for any sequence of point sets, see Roth [18].

[2] If one considers only boxes that are anchored at the origin, i.e. the star-discrepancy, then one can choose $c = \varepsilon_0 = 1/(32e^2) \approx 0.00423$ [8] and $C = 100$ [1].

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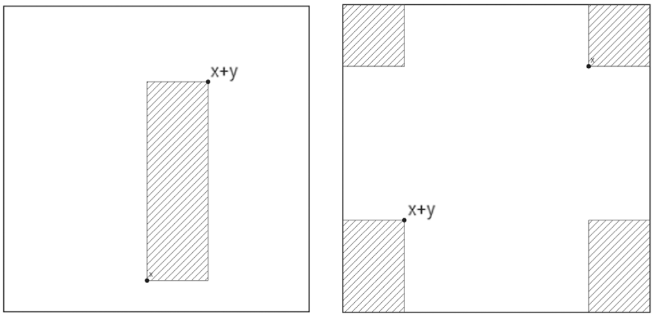

Fig. 1. Two sample test sets from $\mathcal{B}^2_{\text{per}}$.

where

$$B_1(x, y) := (0, y) + x \bmod 1 \tag{2}$$

for $x, y \in [0, 1]^d$. Note that $B_1(x, y)$ is simply $(x, x + y)$ iff $x + y \le 1$. In all other cases one has to respect the geometry of the torus, cf. Fig. 1.

We prove the following theorem.

Theorem 1. *For every $n, d \in \mathbb{N}$ and every point set $\mathcal{P}_n \subset [0, 1]^d$ with $\#\mathcal{P}_n = n$ we have*

$$\mathrm{disp}(\mathcal{P}_n, \mathcal{B}^d_{\text{per}}) \ge \min\{1, d/n\},$$

or equivalently,

$$N_0(\varepsilon, \mathcal{B}^d_{\text{per}}) \ge d/\varepsilon \quad \textit{for } 0 < \varepsilon < 1.$$

Clearly, this implies the following.

Corollary 2. *For every $n, d \in \mathbb{N}$ and every point set $\mathcal{P}_n \subset [0, 1]^d$ with $\#\mathcal{P}_n = n$ we have*

$$D(\mathcal{P}_n, \mathcal{B}^d_{\text{per}}) \ge \min\{1, d/n\}$$

or equivalently,

$$N(\varepsilon, \mathcal{B}^d_{\text{per}}) \ge d/\varepsilon \quad \textit{for } 0 < \varepsilon < 1.$$

As far as we know, the largest lower bound on the inverse of the periodic discrepancy that was known before is due to Hinrichs [8] and states that

$$N(\varepsilon, \mathcal{B}^d_{\text{per}}) \ge N(\varepsilon, \mathcal{B}_*) \ge c\, d/\varepsilon \quad \text{for } 0 < \varepsilon < c,$$

where $\mathcal{B}_*$ is the set of all axis parallel boxes that are anchored at the origin and $c > 0$ can be chosen as $c = 1/(32e^2) \ge 0.004229$. For the proof of this note that $\mathcal{B}^d_{\text{per}} \supset \mathcal{B}_*$.

2. Preliminaries

The main tool for the proof will be the following lemma, which provides a lower bound for the d-dimensional dispersion in terms of the dispersion of certain projections of the point set.

For a set $A \subset [0, 1]^d$ we define the projections

$$A^{(k)} := \left\{(x_1, \ldots, x_k) \in [0, 1]^k : (x_1, \ldots, x_d) \in A\right\}, \quad 1 \le k \le d, \tag{3}$$

i.e. we consider every element from A without the last $d - k$ coordinates. For a family of sets $\mathcal{B} \subset 2^{[0,1]^d}$ we define $\mathcal{B}^{(k)} = \{B^{(k)} : B \in \mathcal{B}\} \subset 2^{[0,1]^k}$.

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Note that, if the ranges satisfy $\mathcal{B} \supset \mathcal{B}^{(k)} \times [0,1]^{d-k} = \{B \times [0,1]^{d-k} : B \in \mathcal{B}^{(k)}\}$ (as $\mathcal{B}^d_{\mathrm{per}}$ or the set of all axis-parallel boxes), then it is obvious that

$$\operatorname{disp}(\mathcal{P}_n, \mathcal{B}) \geq \sup_{\substack{B \in \mathcal{B}^{(k)}: \\ \mathcal{P}_n \cap (B \times [0,1]^{d-k}) = \varnothing}} |B| = \sup_{\substack{B \in \mathcal{B}^{(k)}: \\ \mathcal{P}_n^{(k)} \cap B = \varnothing}} |B| = \operatorname{disp}(\mathcal{P}_n^{(k)}, \mathcal{B}^{(k)})$$

for every point set $\mathcal{P}_n$. The same holds for the discrepancy.

However, such a bound is not sufficient to prove bounds on the dispersion that are growing with the dimension. Hence, we prove a refinement of this inequality. In particular, we need the fact that we can forget about (at least) one point whenever we project to lower dimensions, losing some specific constant. For this, we define

$$\kappa_{\mathcal{B}}(A) := \sup\left\{\kappa \geq 0 \mid \forall \widetilde{B} \in \mathcal{B}^{(d-1)} \; \exists B \in \mathcal{B}: \widetilde{B} = B^{(d-1)},\; A \cap B = \varnothing \text{ and } |B| \geq \kappa |\widetilde{B}|\right\} \tag{4}$$

for ranges $\mathcal{B}$ and a subset $A \subset [0,1]^d$. This constant is to guarantee that for every set $\widetilde{B} \in \mathcal{B}^{(d-1)}$ there is a set $B \in \mathcal{B}$ with volume larger than $\kappa_{\mathcal{B}}(A)\,|\widetilde{B}|$, that is disjoint to A and is projected onto $\widetilde{B}$. Clearly, $\kappa_{\mathcal{B}}(A) \in [0,1]$.

Lemma 3. *For every point set $\mathcal{P} \subset [0,1]^d$ and every $A \subset [0,1]^d$ we have*

$$\operatorname{disp}(\mathcal{P}, \mathcal{B}) \geq \kappa_{\mathcal{B}}(A) \operatorname{disp}\left((\mathcal{P} \setminus A)^{(d-1)}, \mathcal{B}^{(d-1)}\right).$$

Proof. By the definition of $\kappa_{\mathcal{B}}$ we obtain for every $\widetilde{B} \in \mathcal{B}^{(d-1)}$ and $A \subset [0,1]^d$ that

$$\sup\left\{|B| : B \in \mathcal{B} \text{ with } B^{(d-1)} = \widetilde{B} \text{ and } A \cap B = \varnothing\right\} \geq \kappa_{\mathcal{B}}(A)\,|\widetilde{B}|.$$

Now take the supremum over all sets $\widetilde{B}$ with $(\mathcal{P} \setminus A)^{(d-1)} \cap \widetilde{B} = \varnothing$. Clearly, the right hand side then is $\kappa_{\mathcal{B}}(A)$ $\operatorname{disp}\big((\mathcal{P} \setminus A)^{(d-1)}, \mathcal{B}^{(d-1)}\big)$. The left hand side, after taking the supremum, is the supremum of the volumes $|B|$, where B is such that $A \cap B = \varnothing$ and $(\mathcal{P} \setminus A)^{(d-1)} \cap B^{(d-1)} = \varnothing$. The second property implies that $\mathcal{P} \cap B \subset A$ and hence, by the first property, $\mathcal{P} \cap B = \varnothing$. This shows that the left hand side is bounded from above by $\operatorname{disp}(\mathcal{P}, \mathcal{B})$, which proves the statement.

3. Proof of Theorem 1

First we treat the case $n \leq d$. Here, the advantage of the periodic test sets is most clearly observed. Given an arbitrary point set $\mathcal{P}_n = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\} \subset [0,1]^d$ we define $z \in [0,1]^d$ by $z_i = x_i^{(i)}$ for $1 \leq i \leq n$ and $z_i = x_i^{(n)}$ for $n+1 \leq i \leq d$. We consider the test boxes $B_1(z,1)$, cf. (2), which consist of the whole cube without the d hyperplanes $\{x : x_i = z_i\}$, $i = 1, \ldots, d$. Clearly, we obtain $\mathcal{P}_n \cap B_1(z,1) = \varnothing$ and $|B_1(z,1)| = 1$. This proves $\operatorname{disp}(\mathcal{P}_n, \mathcal{B}^d_{\mathrm{per}}) = 1$ whenever $n \leq d$.

We now turn to the case $n > d$. In this case we use Lemma 3. Hence, we have to bound $\kappa(A) := \kappa_{\mathcal{B}^d_{\mathrm{per}}}(A)$, cf. (4), for specific $A \subset [0,1]^d$. In fact, we only need $A = \{t\}$ for some $t = (t_1, \ldots, t_d) \in \mathcal{P}_n$. Note that $(\mathcal{B}^d_{\mathrm{per}})^{(d-1)}$ is the set of all axis-parallel periodic boxes in $[0,1]^{d-1}$, i.e. $(\mathcal{B}^d_{\mathrm{per}})^{(d-1)} = \mathcal{B}^{d-1}_{\mathrm{per}}$. For every $\widetilde{B} \in \mathcal{B}^{d-1}_{\mathrm{per}}$, we have that

$$B := \widetilde{B} \times (0,1) + (0, \ldots, 0, t_d) \bmod 1$$

is an element of $\mathcal{B}^d_{\mathrm{per}}$ that does not contain t, see Fig. 2. Moreover, $|B| = |\widetilde{B}|$. This shows $\kappa(\{t\}) = 1$ for every $t \in [0,1]^d$ and, by Lemma 3,

$$\operatorname{disp}(\mathcal{P}_n, \mathcal{B}^d_{\mathrm{per}}) \geq \operatorname{disp}\left((\mathcal{P}_n \setminus \{t\})^{(d-1)}, \mathcal{B}^{d-1}_{\mathrm{per}}\right).$$

Iterating this procedure another $d-2$ times we obtain

$$\operatorname{disp}(\mathcal{P}_n, \mathcal{B}_{\mathrm{per}}) \geq \operatorname{disp}\left((\mathcal{P}_n \setminus A)^{(1)}, \mathcal{B}^1_{\mathrm{per}}\right)$$

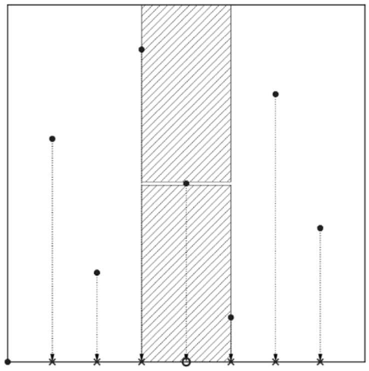

Fig. 2. The set $B = \widetilde{B} \times (0, 1) + (0, \ldots, 0, t_d) \bmod 1$.

for every $A \subset \mathcal{P}_n$ with $\#A = d-1$. Clearly, $\mathcal{B}^1_{\mathrm{per}}$ is the set of all periodic intervals in [0, 1]. After taking the maximum over all A, the latter is the maximal length of a periodic interval that contains at most $d-1$ elements of $\mathcal{P}_n^{(1)}$. This is obviously bounded from below by d/n. This finishes the proof. □

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