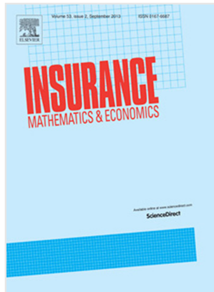

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Optimal Investment Strategy for Participating Contracts

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Abstract

Participating contracts are popular insurance policies, in which the payoff to a policyholder is linked to the performance of a portfolio managed by the insurer. We consider the portfolio selection problem of an insurer that offers participating contracts and has an S-shaped utility function. Applying the martingale approach, closed-form solutions are obtained. The resulting optimal strategies are compared with portfolio insurance hedging strategies (CPPI and OBPI). We also study numerical solutions of the portfolio selection problem with constraints on the portfolio weights.

JEL Classification: C20; C61; G11.

Keywords: Participating contract; utility maximization; martingale and dual approach; concavification technique; stochastic control.

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1. Introduction

We study the continuous time portfolio selection problem for insurance companies managing the portfolios supporting participating insurance contracts. Participating contracts are constructed to allow policyholders to share in the profits of the investment portfolio, while simultaneously receiving a guarantee that limits their downside. The policyholders pay premiums to the insurer and the collected premiums are pooled within the insurance company's *general account*. The contract payoffs are linked to the performance of this account. The insurance company manages the fund in order to hedge its liabilities, and maximize the performance of its residual share of the portfolio after the liabilities have been paid.

The objective of the present paper is to develop optimal asset management strategies for the insurance companies, whereas most of the existing literature focuses either on the pricing aspect of participating contracts or certain characterization of the risk which the insurance companies are exposed to from writing these contracts. For example, Briys and De Varenne [4] derive a closed-form valuation based on an option pricing approach for the participating contract, where the policyholder receives a guaranteed rate of interest (namely *point-to-point basis guarantee*) and some bonuses determined as a fraction of financial gains at the maturity of the contract. Other work on pricing includes Grosen and Jørgensen [13], Siu [22], and Fard and Siu [9]. The literature that focuses on the characterization of insurance companies' risk exposure includes Kling et al. [18], Gatzert and Kling [12], and Bernard and Le Courtois [2], among others. Kling et al. [18], and Gatzert and Kling [12] investigate some standard risk measures of the participating contracts known as *cliquet-style guarantees*, for which the policyholder is credited with a certain rate of return every year. Bernard and Le Courtois [2] study the resulting risk profile of both the insurance company and policyholders under two well-known portfolio insurance strategies (i.e., CPPI and OBPI). Earlier work on asset and liability management for participating contracts has often focused on the problem in discrete time with a finite scenario set. The advantage of this setting is that it allows one to consider more complex and flexible contract structures. Its disadvantages include a lack of closed form solutions, and computational challenges in generating and working with scenario trees. Examples include Consiglio et al. [7] and Consiglio et al. [6], both of which employ scenario optimization in discrete time to analyze problems faced by insurers offering participating contracts with minimum guarantees. For a general stochastic control formulation of the problem facing an insurer maximizing expected utility of the surplus of assets net of liabilities, see Rudolf and Ziemba [21].

Utility based portfolio selection problems have been intensively studied in the literature on mathematical finance and economics; see, for example, Cvitanić and Karatzas [8], Karatzas et al. [17] and Karatzas and Shreve [16]. Our problem differs due to the inclusion of a liability consisting of a participating contract in the investment portfolio. Moreover, decision-makers are taken to be risk averse with respect to gains and risk seeking with respect to losses, which results in a *S-shaped* power utility function. This utility function is exploited in our problem to reflect this behavioral perspective for the insurance company, which plays the role of the asset manager, to derive explicit optimal investment strategies for two participating contracts with *point-to-point basis guarantees*, which we call (following Bernard et al. [3]) *the defaultable participating contract* and *the fully protected participating contract.* The solutions provide insights for the insurance company in constructing portfolios to serve its purposes.

Our derivation of the optimal solutions relies on a combination of a martingale approach and a pointwise optimization technique. The legitimacy of the martingale approach follows from the completeness of the market model we consider. The approach entails determining the best terminal portfolio value and recovering the dynamic investment strategies from this payoff. In the pointwise optimization procedure, we adopt a concavification technique, which has been used by Carpenter [5] and later by He and Kou [14].

As we previously noted, one payoff function we consider in this paper is based on a *point-to-point basis guarantee*, following Briys and De Varenne [4], and its shape is similar to that of the first-loss fee scheme for hedge funds studied by He and Kou [14]. However, in our problem the positive payoff for the insurance company consists of two pieces with a kink point, while in He and Kou [14] the positive part of payoff is smooth without any kink. Therefore, the use of an *S-shaped* utility function in our problem sets results in an objective function different from that considered by He and Kou [14]. Moreover He and Kou [14] consider a liquidation barrier for the fund. When the portfolio drops below this boundary, the fund is liquidated immediately. In contrast, we do not employ a liquidation barrier. These problem characteristics significantly complicate the analysis, and the final form of the optimal solutions.

The completeness of the financial market is a key assumption for our derivation of explicit optimal solutions by the martingale approach. In practice, however, regulatory requirements aimed at controlling solvency risk may prevent the insurance company from investing more than a certain fraction of total wealth in the risky assets. In the presence of such regulatory restrictions, the market is no longer complete for the insurance company, and analytical solutions of the control problem are in general no longer attainable. In this paper, we resort to a numerical procedure to compute the optimal solutions in the constrained case to facilitate comparison with the solutions derived by the martingale approach for the unconstrained case.

The remainder of the paper is structured as follows. Section 2 describes participating contracts and presents the formulation of the stochastic control problem. Auxiliary problem formulations are also given in this section. In Section 3, we solve the auxiliary problems using Lagrangian duality and the pointwise optimization technique. The justification for the concavification technique is included in this section as well. Section 4 presents the optimal portfolio value processes and optimal trading strategies for the stochastic control problems. Section 5 presents numerical examples for the solutions from Section 4. In Section 6, we consider the constrained portfolio problem with bounded control. The last section provides further discussion and concludes the paper.

2. Participating Contracts and Problem Formulation

2.1. Basics of participating contracts

Let L_0 be the policyholder's total contribution and α be the initial liability-to-asset ratio of the insurer so that the initial capital in the insurer's *general account* is $x_0 := L_0/\alpha > 0$.

We assume that the capital in the general account is invested in a risky asset S and a risk-free bond B with price processes as follows:

$$\begin{cases} dB_t = rB_t dt, \\ dS_t = \mu S_t dt + \sigma S_t dW_t, \end{cases}$$

where r is the risk-free rate, $\mu > r$ is the growth rate of the risky asset, $\sigma > 0$ is the volatility, and $W := \{W_t, t \geq 0\}$ is a standard Brownian motion under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbf{F} := \{\mathcal{F}_t, t \geq 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \leq s \leq t)$ of the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let π_t denote the amount of capital invested in the risky asset S at time t, $t \geq 0$. With a trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$, the total portfolio value process, denoted by X_t^π, evolves as follows:

$$dX_t^\pi = [rX_t^\pi + \pi_t(\mu - r)]dt + \sigma\pi_t dW_t. \tag{1}$$

It is natural to assume that the trading strategy π is $\mathbf{F}$-progressively measurable and satisfies $\int_0^T \pi_t^2 dt < \infty$ a.s., which guarantees the existence and uniqueness of strong solution to (1).

The terminal portfolio value X_T^π is shared between the policyholder and the insurer according to a pre-described scheme with certain guarantee features in favor of the policyholder. Below, we introduce two participating contracts with terminal guarantees: (1) a *defaultable participating contract*; and (2) a *fully protected participating contract*. In both contracts, the policyholder is guaranteed a minimum growth rate g (see Briys and De Varenne [4]) and the guaranteed amount at maturity time T is $L_T^g = L_0 e^{gT}$, where L_0 is the initial liability of the insurer. g is set lower than the risk-free rate.

In the defaultable participating contract, the payoff to the policyholder is given as follows:

$$\Theta(X_T^\pi) = L_T^g + \delta(\alpha X_T^\pi - L_T^g)_+ - (L_T^g - X_T)_+ = \begin{cases} X_T^\pi, & X_T^\pi < L_T^g, \\ L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ \delta\alpha X_T^\pi + (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}, \end{cases} \tag{2}$$

where $(x)_+ = \max\{x, 0\}$ for a real number x and the liability-to-asset ratio $\alpha \in (0,1)$. The payoff for the policyholder is equal to the guaranteed amount L_T^g, plus a scaled long potion in a call option and a short position in a put. When the terminal portfolio value is less than the guaranteed amount L_T^g, the contract 'defaults', and the policyholder only receives the portfolio value as payoff. With the amount of $\Theta(X_T^\pi)$ paid to the policyholder, the insurer retains a payoff as follows

$$\Psi(X_T^\pi) = X_T^\pi - \Theta(X_T^\pi) = \begin{cases} 0, & X_T^\pi < L_T^g, \\ X_T^\pi - L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ (1-\delta\alpha)X_T^\pi - (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}, \end{cases} \tag{3}$$

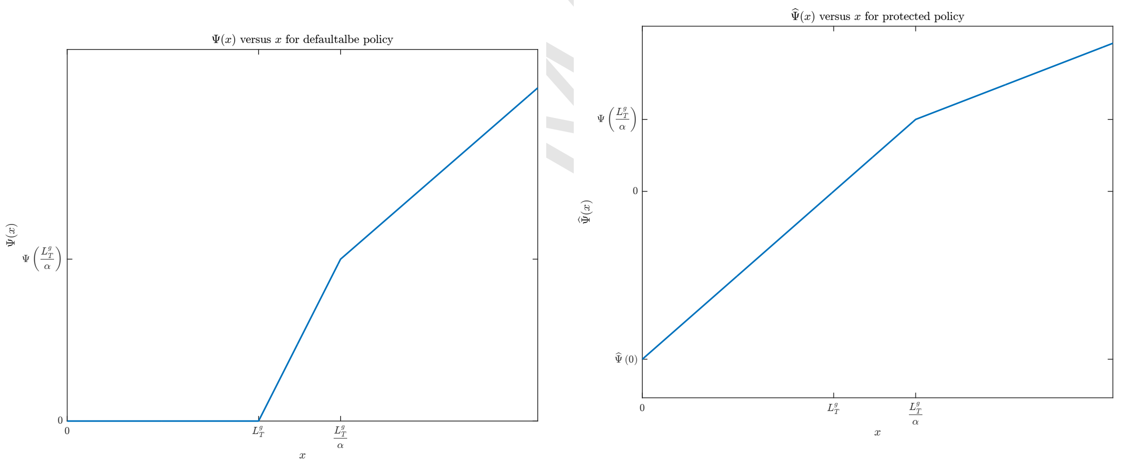
Note that for the defaultable policy, the payoff of the policyholder is not really guaranteed at L_T^g. Instead, when the terminal portfolio value X_T^π is smaller than the guaranteed amount, the policyholder is only entitled to the portfolio value. In contrast, following the work by Bernard et al. [3], we also investigate the fully protected participating contract that entitles the policyholder to a payoff as follows:

$$\widehat{\Theta}(X_T^\pi) = L_T^g + \delta(\alpha X_T^\pi - L_T^g)_+ = \begin{cases} L_T^g, & X_T^\pi < L_T^g, \\ L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ \delta\alpha X_T^\pi + (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}, \end{cases} \tag{4}$$

which differs from the payoff structure in equation (2) only in the first case where $X_T < L_T^g$. Correspondingly, the payoff of the insurer becomes

$$\widehat{\Psi}(X_T^\pi) = X_T^\pi - \widehat{\Theta}(X_T^\pi) = \begin{cases} X_T^\pi - L_T^g, & X_T^\pi < L_T^g, \\ X_T^\pi - L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ (1-\delta\alpha)X_T^\pi - (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}. \end{cases} \tag{5}$$

While the worst payoff to the insurer in the defaultable contract is zero, the payoff could be negative for the fully protected contract, which occurs whenever the portfolio value becomes less than the guaranteed amount L_T^g. The payoff curves for both policies are illustrated in Figure 1.



(a) Insurer's payoff versus terminal portfolio value x for the defaultable participating contract.

(b) Insurer's payoff versus terminal portfolio value x for the fully protected participating contract.

Figure 1: Insurer's payoff for the two participating contracts.

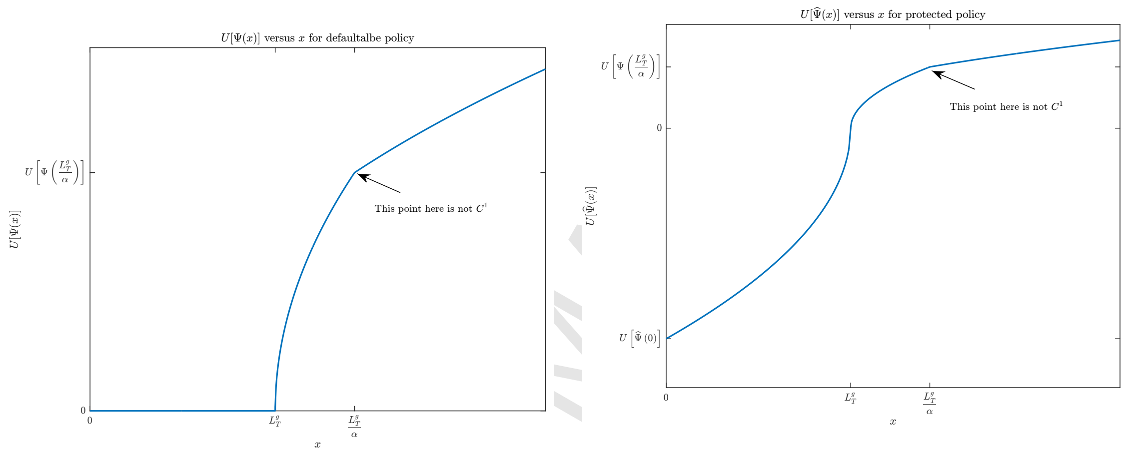
2.2. *Problem formulation*

We formulate the decision of the insurer as an expected utility maximization problem with an S-shaped utility function from prospect theory, for which decision-makers are risk averse with respect to gains and risk seeking with respect to losses. More specifically, the utility function is continuous, and increasing, concave on $[0, \infty)$, and convex on $(-\infty, 0]$ and assumes the following form:

$$U(x) = \begin{cases} x^{\gamma}, & x \geq 0, \\ -\lambda(-x)^{\gamma}, & x < 0, \end{cases} \tag{6}$$

where $0 < \gamma < 1$ measures the degree of risk aversion from gain and risk seeking when loss occurs. The parameter $\lambda > 1$ is called loss aversion degree, and it measures the extent to which individuals are *loss averse*, see Tversky and Kahneman [23].

The functions $U[\Psi(x)]$ and $U[\widehat{\Psi}(x)]$ are depicted in Figure 2.



(a) Insurer's utility level versus terminal portfolio value x for the defaultable participating contract.

(b) Insurer's utility level versus terminal portfolio value x for the fully protected participating contract.

Figure 2: Insurer's utility level for the two participating contracts.

Definition 2.1. *A trading strategy* $\pi := \{\pi_t, 0 \leq t \leq T\}$ *is called admissible with initial wealth* $x_0 > 0$ *if it belongs to the following set:*

$$\mathcal{A}(x_0) := \{\pi \in \mathcal{S} : \ X_0^{\pi} = x_0 \ \text{ and } \ X_t^{\pi} \geq 0, a.s., \forall\, 0 \leq t \leq T\}, \tag{7}$$

where $\mathcal{S}$ *denotes the set of* $\mathbf{F}$*-progressively measurable processes* π *such that* $\int_0^T \pi_t^2 dt < \infty$ *a.s.*

To proceed, we define the the market price of risk, i.e. "relative risk", as

$$\zeta := \frac{\mu - r}{\sigma},$$

and the price density process as

$$\xi_t := \exp\left\{-\left(r+\frac{\zeta^2}{2}\right)t-\zeta W_t\right\}. \tag{8}$$

Further, for $t \le s$, we define

$$\xi_{t,s} = \xi_t^{-1}\xi_s = \exp\left[-(r+\frac{\zeta^2}{2})(s-t)-\zeta(W_s-W_t)\right], \tag{9}$$

which is independent of $\mathcal{F}_t$. Note that $\xi_t = \xi_{0,t}$.

We apply Itô's formula in conjunction with equations (1) and (8) to obtain

$$\xi_t X_t^\pi = x_0 + \int_0^t \xi_s(\sigma\pi_s - \zeta X_s^\pi)dW_s, \;\; t \in [0,T]. \tag{10}$$

The right-hand side is a non-negative local martingale and thus a super-martingale, which implies $\mathbb{E}[\xi_T X_T^\pi] \le x_0$; see Proposition 1.1.7 in Pham [19] or Chapter 1, Problem 5.19 in Karatzas and Shreve [15]. As a consequence, we formulate the insurer's optimal investment decision for the two participating contracts as follows:

- Defaultable participating insurance contract:

$$\begin{cases} \sup\limits_{\pi\in\mathcal{A}(x_0)} & \mathbb{E}[U(\Psi(X_T^\pi))], \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \le x_0. \end{cases} \tag{11}$$

- Fully protected participating insurance contract:

$$\begin{cases} \sup\limits_{\pi\in\mathcal{A}(x_0)} & \mathbb{E}\left[U(\widehat{\Psi}(X_T^\pi))\right], \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \le x_0. \end{cases} \tag{12}$$

Since the payoff $\Psi(X_T^\pi)$ is non-negative in every state, the *S-shaped* utility is the same as a power utility $U(x) = x^\gamma$, $x \ge 0$, for problem (11). In contrast, for the fully protected participating contract, the insurer may suffer from a loss and therefore, the negative part of the *S-shaped* utility $U(\cdot)$ does play a role in problem (12).

2.3. Auxiliary problems

We will adopt a martingale approach to solve problems (11) and (12). Let $\mathcal{M}_+$ denote the set of non-negative $\mathcal{F}_T$-measurable random variables, and consider the following two auxiliary problems:

$$\begin{cases} \sup\limits_{Z\in\mathcal{M}_+} & \mathbb{E}[U(\Psi(Z))], \\ \text{subject to} & \mathbb{E}[\xi_T Z] \le x_0, \end{cases} \tag{13}$$

and

$$\begin{cases} \sup_{Z \in \mathcal{M}_+} & \mathbb{E}\left[U(\widehat{\Psi}(Z))\right], \\ \text{subject to} & \mathbb{E}[\xi_T Z] \leq x_0. \end{cases} \tag{14}$$

An optimal solution can be obtained for each of these two auxiliary problems such that the constraint is binding at the solution; see Lemma 3.2 and Proposition 3.5 in section 3.

From the solutions of auxiliary problems, we can construct optimal trading strategies for problems (11) and (12) as explained below. Let Z^* and $\widehat{Z}$ respectively denote optimal solutions to the above two problems with $\mathbb{E}[\xi_T Z^*] = \mathbb{E}[\xi_T \widehat{Z}] = x_0$, and define

$$Y_t^* := \xi_t^{-1}\mathbb{E}[\xi_T Z^* | \mathcal{F}_t] \text{ and } \widehat{Y}_t := \xi_t^{-1}\mathbb{E}[\xi_T \widehat{Z} | \mathcal{F}_t], \ 0 \leq t \leq T. \tag{15}$$

Obviously, both $\{\xi_t Y_t^*, 0 \leq t \leq T\}$ and $\{\xi_t \widehat{Y}_t, 0 \leq t \leq T\}$ are $\mathbf{F}$-martingales under $\mathbb{P}$. Thus, they admit the following representation by the martingale representation theorem (see Chapter 3, Theorem 4.15 and Problem 4.16 in Karatzas and Shreve [15] or Theorem 1.2.9 in Pham [19]):

$$\xi_t Y_t^* = x_0 + \int_0^t \theta_s^* dW_s \text{ and } \xi_t \widehat{Y}_t = x_0 + \int_0^t \widehat{\theta}_s dW_s, \ 0 \leq t \leq T, \tag{16}$$

for some $\mathbb{R}$-valued $\mathcal{F}_t$-progressively measurable processes $\{\theta_t^*, 0 \leq t \leq T\}$ and $\{\widehat{\theta}_t, 0 \leq t \leq T\}$ satisfying $\int_0^T (\theta_t^*)^2 dt < \infty$ and $\int_0^T (\widehat{\theta}_t)^2 dt < \infty$, a.s. In particular, both $\{\xi_t Y_t^*, 0 \leq t \leq T\}$ and $\{\xi_t \widehat{Y}_t, 0 \leq t \leq T\}$ are continuous, a.s.

Proposition 2.1. *Let Z^* and $\widehat{Z}$ respectively denote optimal solutions to problems* (13) *and* (14)*. For the two processes $\{\theta_t^*, 0 \leq t \leq T\}$ and $\{\widehat{\theta}_t, 0 \leq t \leq T\}$ given in equation* (16)*, define*

$$\pi_t^* = \sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^* \ \text{ and } \ \widehat{\pi}_t = \sigma^{-1}\xi_t^{-1}\widehat{\theta}_t + \sigma^{-1}\zeta \widehat{Y}_t. \tag{17}$$

Then, $\pi^* := \{\pi_t^*, 0 \leq t \leq T\} \in \mathcal{A}(x_0)$ and $\widehat{\pi} := \{\widehat{\pi}_t, 0 \leq t \leq T\} \in \mathcal{A}(x_0)$ solve problems (11) *and* (12)*, respectively, and the optimal portfolio values at time t, $0 \leq t \leq T$, are given by $X_t^{\pi^*} = Y_t^*$ and $X_t^{\widehat{\pi}} = \widehat{Y}_t$ for the two problems, respectively.*

Proof. We only show the properties of π^* for problem (11), because the result follows in parallel for $\widehat{\pi}$. From expressions (15) and (16),

$$d\left(\xi_t Y_t^*\right) = \theta_t^* dW_t, \ Y_0^* = x_0, \ \text{and } Y_T^* = Z^*, \ \text{a.s.} \tag{18}$$

where the price density process ξ_t is defined in (8) satisfying $d\xi_t^{-1} = \xi_t^{-1}\left[(r+\zeta^2)dt + \zeta dW_t\right]$. Therefore, applying the Itô product rule yields

$$\begin{aligned} dY_t^* &= \xi_t^{-1} d\xi_t Y_t^* + \xi_t Y_t^* d\xi_t^{-1} + d\xi_t^{-1} d\xi_t Y_t^* \\ &= \left[Y_t^*(r+\zeta^2) + \xi_t^{-1}\zeta\theta_t^*\right] dt + \left[\xi_t^{-1}\theta_t^* + Y_t^*\zeta\right] dW_t \\ &= [rY_t^* + \pi_t^*(\mu - r)]dt + \sigma\pi_t^* dW_t, \end{aligned} \tag{19}$$

where the last step follows from (17).

Since $\int_0^T (\theta_t^*)^2 dt < \infty$ a.s., the stochastic differential equation (SDE) (19) admits

$$Y_t^* = \xi_t^{-1}\mathbb{E}[\xi_T Z^* | \mathcal{F}_t]$$

as its unique solution which is continuous almost surely. The SDE (19) agrees with (1). Thus, by the uniqueness of strong solutions, we have $\mathbb{P}\left(X_t^{\pi^*} = Y_t^*,\ \ t \in [0,T]\right) = 1$. In addition, it is obvious that $X_t^{\pi^*} = Y_t^* \geq 0$ a.s., $t \in [0,T]$.

Moreover,

$$\begin{aligned} &\int_0^T (\pi_t^*)^2\, dt = \int_0^T \left(\sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^*\right)^2 dt \\ \leq& 2\sigma^{-2} \cdot \max_{0\leq t\leq T} |\xi_t^{-2}| \cdot \int_0^T (\theta_t^*)^2\, dt + 2\sigma^{-2}\zeta^2 T \cdot \max_{0\leq t\leq T} \left| (Y_t^*)^2 \right| < \infty, \ \text{ a.s.}, \end{aligned}$$

where we use the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and the fact that $\{\xi_t, \forall 0 \leq t \leq T\}$ is a strictly positive process and the almost sure continuity of both $\{\xi_t^{-2}, \forall 0 \leq t \leq T\}$ and $\{(Y_t^*)^2, \forall 0 \leq t \leq T\}$. Therefore, $\pi^* \in \mathcal{A}(x_0)$.

On the other hand, any X_T^π is $\mathcal{F}_T$-measurable and thus, $X_T^\pi \in \mathcal{M}_+$, $\forall \pi \in \mathcal{A}(x_0)$. Consequently, the optimality of Z^* for problem (13) implies

$$\mathbb{E}[U(\Psi(X_T^{\pi^*}))] = \mathbb{E}[U(\Psi(Z^*))] \geq \mathbb{E}[U(\Psi(X_T^\pi))], \ \ \forall \pi \in \mathcal{A}(x_0),$$

which means that π^* solves problem (11). The claim about the optimal portfolio value follows immediately. □

3. Optimal Solutions to Auxiliary Problems

The analysis in the last section motivates us to focus on the two auxiliary problems (13) and (14). Once we solve these problems, we can find θ_s^* and $\widehat{\theta}_s$ via equations (15) and (16) and eventually apply Proposition 2.1 to derive the optimal trading strategies π^* and $\widehat{\pi}$.

3.1. Lagrangian duality problems and pointwise optimization problems

We solve the two auxiliary problems (13) and (14) by a Lagrangian duality method and show that an optimal solution can be obtained such that the constraint is binding at the solution. This entails introducing the following Lagrange dual problems with multipliers β and ν:

$$\sup_{Z\in\mathcal{M}_+} \quad \mathbb{E}[U(\Psi(Z)) - \beta\xi_T Z], \;\; \beta > 0, \tag{20}$$

and

$$\sup_{Z\in\mathcal{M}_+} \quad \mathbb{E}\left[U(\widehat{\Psi}(Z)) - \nu\xi_T Z\right], \;\; \nu > 0. \tag{21}$$

To study the above problems, we resort to a pointwise optimization procedure which involves solving the following two problems indexed by $y > 0$:

$$\sup_{x\in\mathbb{R}_+} \quad \left[U(\Psi(x)) - yx\right], \tag{22}$$

and

$$\sup_{x\in\mathbb{R}_+} \quad \left[U(\widehat{\Psi}(x)) - yx\right], \tag{23}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Lemma 3.1. *Let $x^*(y)$ and $\widehat{x}(y)$ be two Borel measurable functions $x^*(y)$ solves* (22) *and $\widehat{x}(y)$ solves* (23) *for each $y > 0$. Define*

$$Z^*_\beta := x^*(\beta\xi_T) \;\; \text{and} \;\; \widehat{Z}_\nu := \widehat{x}(\nu\xi_T).$$

Then, Z^*_β and $\widehat{Z}_\nu$ solve problems (20) *and* (21) *respectively.*

Proof. We only show the optimality of Z^*_β. Indeed, we obviously have $Z^*_\beta \in \mathcal{M}_+$, and moreover, by the optimality of the function $x^*(y)$ for problem (22), for any $Z \in \mathcal{M}_+$ and $\beta > 0$ we obtain

$$\begin{aligned}
\mathbb{E}[U(\Psi(Z)) - \beta\xi_T Z] &= \int [U(\Psi(Z)) - \beta\xi_T Z]\, d\mathbb{P} \\
&\le \int [U(\Psi(x^*(\beta\xi_T))) - \beta\xi_T x^*(\beta\xi_T)]\, d\mathbb{P} \\
&= \int \left[U\left(\Psi(Z^*_\beta)\right) - \beta\xi_T Z^*_\beta\right] d\mathbb{P} \\
&= \mathbb{E}[U(\Psi(Z^*_\beta)) - \beta\xi_T Z^*_\beta],
\end{aligned}$$

by which the proof is complete. □

Lemma 3.2. *(a) Assume that there exists a constant $\beta^* > 0$ such that $Z^*_{\beta^*} \in \mathcal{M}_+$ solves* (20) *with $\beta = \beta^*$ and $\mathbb{E}[\xi_T Z^*_{\beta^*}] = x_0$. Then, $Z^* := Z^*_{\beta^*}$ solves problem* (13).

(b) Assume that there exists a constant $\widehat{\nu} > 0$ such that $\widehat{Z}_{\widehat{\nu}} \in \mathcal{M}_+$ solves (21) *with $\nu = \widehat{\nu}$ and $\mathbb{E}[\xi_T \widehat{Z}_{\widehat{\nu}}] = x_0$. Then, $\widehat{Z} := \widehat{Z}_{\widehat{\nu}}$ solves problem* (14).

Proof. We only show part (a). Let $v(x_0)$ denote the supreme value of problem (13) with initial wealth x_0. Then, it follows

$$\begin{aligned} v(x_0) &= \sup_{\substack{Z\in\mathcal{M}_+ \\ \mathbb{E}[\xi_T Z]\le x_0}} \mathbb{E}[U(\Psi(Z))] = \sup_{\substack{Z\in\mathcal{M}_+ \\ \mathbb{E}[\xi_T Z]\le x_0}} \{\mathbb{E}[U(\Psi(Z))] + \beta^* (\mathbb{E}[x_0 - \xi_T Z])\} \\ &\le \sup_{Z\in\mathcal{M}_+} \{\mathbb{E}[U(\Psi(Z))] + \beta^* (\mathbb{E}[x_0 - \xi_T Z])\} \\ &= \mathbb{E}[U(\Psi(Z^*_{\beta^*}))] - \beta^* \left(\mathbb{E}[\xi_T Z^*_{\beta^*}] - x_0\right) \\ &= \mathbb{E}[U(\Psi(Z^*_{\beta^*}))] \le v(x_0), \end{aligned}$$

where the last step is due to the fact that Z_{β^*} is feasible for problem (13). Hence, $Z^* \equiv Z^*_{\beta^*}$ solves problem (13). □

3.2. Solutions of the pointwise optimization problems

The payoff structures for the defaultable and protected policies, $\Psi(x)$ and $\widehat{\Psi}(x)$, are given in (3) and (5). With $U(\cdot)$ given by (6), $U[\Psi(x)]$ zero for $x \leqslant L_g^T$, and concave for $x \geqslant L_g^T$, while $U[\widehat{\Psi}(x)]$ is convex when $x < L_T^g$ and concave for $x \geq L_T^g$. The utility of the insurance company's payoff in each case is illustrated in Figure 2.

We employ the concavification technique from Carpenter [5] (see also He and Kou [14]) to find optimal solutions of problems (22) and (23). We denote the concave envelope of a function f with domain D by f^c.

$$f^c(x) := \inf\{g(x) : D \to \mathbb{R} \mid g(t) \text{ is a concave function}, \;\; g(t) \geq f(t), \;\; \forall t \in D\}, \;\; x \in D$$

We consider the following concavificated versions of problems (22) and (23):

$$\sup_{x\in\mathbb{R}_+} \left[(U \circ \Psi)^c(x) - yx\right], \;\; y > 0, \tag{24}$$

and

$$\sup_{x\in\mathbb{R}_+} \left[(U \circ \widehat{\Psi})^c(x) - yx\right], \;\; y > 0. \tag{25}$$

Proposition 3.3. *For each* $y > 0$*, let* $x^*(y)$ *and* $\widehat{x}(y)$ *be solutions to problems* (24) *and* (25)*, respectively. If* $(U \circ \Psi)^c(x^*(y)) = (U \circ \Psi)(x^*(y))$ *and* $(U \circ \widehat{\Psi})^c(\widehat{x}(y)) = (U \circ \widehat{\Psi})(\widehat{x}(y))$*, then* $x^*(y)$ *and* $\widehat{x}(y)$ *solve the problems* (22) *and* (23)*, respectively.*

Proof. We only show the property of $x^*(y)$. Given $y > 0$, $\forall x \in \mathbb{R}_+$, we have

$$(U \circ \Psi)(x^*(y)) - y \cdot x^*(y) = (U \circ \Psi)^c(x^*(y)) - y \cdot x^*(y) \geqslant (U \circ \Psi)^c(x) - yx \geqslant (U \circ \Psi)(x) - yx.$$

□

The derivation of solutions for the above problems employs the one-sided derivatives of $G(x) = U(\Psi(x))$ and $\widehat{G}(x) = U\left[\widehat{\Psi}(x)\right]$ at $x = \alpha^{-1}L_T^g$. It is easy to verify that $m := G'_-(\alpha^{-1}L_T^g) = \widehat{G}'_-(\alpha^{-1}L_T^g) = \gamma(\alpha^{-1}L_T^g - L_T^g)^{\gamma-1}$, and $G'_+(\alpha^{-1}L_T^g) = \widehat{G}'_+(\alpha^{-1}L_T^g) = (1-\delta\alpha)m$.

Proposition 3.4. *(a) The following function* $x^*(y)$ *solves both problems* (22) *and* (24)*:*

Case A1*. If* $1 - \alpha > \gamma$*, then*

$$x^*(y) = f_1(y; \tilde{z}, k) := \begin{cases} \dfrac{\left[\frac{y}{\gamma(1-\delta\alpha)}\right]^{\frac{1}{\gamma-1}} + (1-\delta)L_T^g}{1-\delta\alpha}, & 0 < y < (1-\delta\alpha)m, \\ \frac{L_T^g}{\alpha}, & (1-\delta\alpha)m \le y \le m, \\ \left(\dfrac{y}{\gamma}\right)^{\frac{1}{\gamma-1}} + L_T^g, & m < y < k, \\ 0, & y \ge k, \end{cases} \tag{26}$$

where $\tilde{z} = \frac{L_T^g}{1-\gamma}$ *and* $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1}$.

Case A2*. If* $(1-\delta\alpha)\gamma > 1 - \alpha$*, then*

$$x^*(y) = f_2(y; \tilde{z}, k) := \begin{cases} \dfrac{\left[\frac{y}{\gamma(1-\delta\alpha)}\right]^{\frac{1}{\gamma-1}} + (1-\delta)L_T^g}{1-\delta\alpha}, & 0 < y < k, \\ 0, & y \ge k, \end{cases} \tag{27}$$

where $\tilde{z} = \frac{(1-\delta)L_T^g}{(1-\delta\alpha)(1-\gamma)}$ *and* $k = \gamma(1-\delta\alpha)[(1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g]^{\gamma-1}$.

Case A3. *If $\gamma \geq 1-\alpha \geq (1-\delta\alpha)\gamma$, then*

$$x^*(y) = f_3(y;\tilde{z},k) := \begin{cases} \dfrac{\left[\frac{y}{\gamma(1-\delta\alpha)}\right]^{\frac{1}{\gamma-1}} + (1-\delta)L_T^g}{1-\delta\alpha}, & 0 < y < (1-\delta\alpha)m, \\ \tilde{z}, & (1-\delta\alpha)m \leq y < k, \\ 0, & y \geq k, \end{cases} \tag{28}$$

where $\tilde{z} = \frac{L_T^g}{\alpha}$ and $k = (1-\alpha)^\gamma (\tilde{z})^{\gamma-1}$.

(b) The following function $\widehat{x}(y)$ solves both problems (23) and (25):

Case B1. *If $\lambda > \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then there exists a unique solution $\tilde{z} \in (L_T^g, \frac{L_T^g}{\alpha})$ satisfying*

$$[(\gamma-1)\tilde{z} + L_T^g](\tilde{z} - L_T^g)^{\gamma-1} - \lambda(L_T^g)^\gamma = 0. \tag{29}$$

The optimal solution is given by $\widehat{x}(y) = f_1(y;\tilde{z},k)$, where $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1}$ and the function $f_1(y;\tilde{z},k)$ is defined in (26).

Case B2. *If $\lambda < \frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then there exists a unique solution $\tilde{z} \in (\frac{L_T^g}{\alpha}, \infty)$ of*

$$[(1-\delta\alpha)(\gamma-1)\tilde{z} + (1-\delta)L_T^g] \times [(1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g]^{\gamma-1} - \lambda(L_T^g)^\gamma = 0. \tag{30}$$

The optimal solution is given by $\widehat{x}(y) = f_2(y;\tilde{z},k)$, where $k = \gamma(1-\delta\alpha)[(1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g]^{\gamma-1}$ and the function $f_2(y;\tilde{z},k)$ is defined in (27).

Case B3. *If $\frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1} \leq \lambda \leq \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then the optimal solution $\widehat{x}(y) = f_3(y;\tilde{z},k)$ with $\tilde{z} = \frac{L_T^g}{\alpha}$ and $k = \alpha\left[\left(\frac{1-\alpha}{\alpha}\right)^\gamma + \lambda\right](L_T^g)^{\gamma-1}$, where the function $f_3(y;\tilde{z},k)$ is defined in (28).*

Proof. The concave envelopes of $U(\Psi(x))$ and $U\left[\widehat{\Psi}(x)\right]$ are given in Lemmas A.2 and A.3 in Appendix A. To find a maximizer of $h(x) := (U \circ \Psi)^c(x) - yx$, for a given y, one then simply needs to find the points $x^*(y)$ for which 0 is in the superdifferential of h, which is determined by straightforward calculation. Then, observing that $(U \circ \Psi)(x) = (U \circ \Psi)^c(x)$ when $x \in \{0\} \cup [\tilde{z}, \infty)$ and that $x^*(y) \in \{0\} \cup [\tilde{z}, \infty) \subseteq \{U \circ \Psi = (U \circ \Psi)^c\}$ yields the result in part(a). The results of part(b) follow in the same manner. □

3.3. Derivation of the solutions to auxiliary problems (13) *and* (14)

For each $\beta > 0$, define $Z^*_\beta := x^*(\beta\xi_T)$ with function x^* given in equations (26), (27) and (28) for the three distinct cases respectively. Then, combining Lemma 3.1 and Proposition 3.4, Z^*_β solves problem (20). Similarly, for each $\nu > 0$, define $\widehat{Z}_\nu := \widehat{x}(\nu\xi_T)$ where the function $\widehat{x}$ is given in part (b) of Proposition 3.4. Then, $\widehat{Z}_\nu$ solves problem (21). Consequently, by Lemma 3.2, if there exists a nonnegative constant β^* satisfying $\mathbb{E}[\xi_T x^*(\beta^*\xi_T)] = x_0$, then $Z^* = Z^*_{\beta^*}$ solves the auxiliary problem (13). Similarly, if there exists a nonnegative constant $\widehat{\nu}$ satisfying $\mathbb{E}[\xi_T\widehat{x}(\widehat{\nu}\xi_T)] = x_0$, then $\widehat{Z} := \widehat{Z}_{\widehat{\nu}}$ solves problem (14). Proposition 3.5 below guarantees the existence of such $\beta^* > 0$ and $\widehat{\nu} > 0$.

We use Φ and ϕ to denote the standard normal distribution function and its density function. Further, define

$$\begin{cases} d_{1,t}(\beta) := \dfrac{\ln\beta - \ln\xi_t + (r - \frac{1}{2}\zeta^2)(T-t)}{\zeta\sqrt{T-t}}, \\ d_{2,t}(\beta) := d_{1,t}(\beta) + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}, \\ K(\beta) := \phi[d_{1,t}(\beta)]\left(1 + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}\dfrac{\Phi[d_{2,t}(\beta)]}{\phi[d_{2,t}(\beta)]}\right). \end{cases} \tag{31}$$

Proposition 3.5. *(a) There exists a constant $\beta^* > 0$ such that $Z^*_{\beta^*} := x^*(\beta^*\xi_T)$ and $\mathbb{E}[\xi_T Z^*_{\beta^*}] = x_0$, where the function x^* is given in part (a) of Proposition 3.4.*

(b) There exists a constant $\widehat{\nu} > 0$ such that $\widehat{Z}_{\widehat{\nu}} := \widehat{x}(\widehat{\nu}\xi_T)$ and $\mathbb{E}[\xi_T\widehat{Z}_{\widehat{\nu}}] = x_0$, where the function $\widehat{x}$ is given in part (b) of Proposition 3.4.

Proof. We only prove part (a), because part (b) can be proved in a similar way. Define $H(\beta) = \mathbb{E}[\xi_T Z^*_\beta] \equiv \mathbb{E}[\xi_T x^*(\beta\xi_T)]$. We first show the continuity of $H(\beta)$ with respect to β for $\beta > 0$. As shown in (26), (27) and (28), for each of the three Cases A1, A2, and A3, we can write $x^*(\cdot)$ as a piecewise function such that $x^*(\beta\xi_T)$ is the summation of $c_1\mathbf{1}_{\{\beta\xi_T\le c_2\}}$ and $c_3(\beta\xi_T)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\beta\xi_T\le c_4\}}$ with appropriate choices of non-negative constants c_1, c_2, c_3 and c_4. We can then use the formula in Appendix B to obtain

$$\begin{cases} c_1\mathbb{E}\left[\xi_T\mathbf{1}_{\{\beta\xi_T\le c_2\}}\right] = c_1 e^{-rT}\Phi[d_{1,0}(c_2/\beta)], \\ c_3\mathbb{E}\left[\xi_T(\beta\xi_T)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\beta\xi_T\le c_4\}}\right] = c_3 e^{-rT}\frac{\phi[d_{1,0}(1/\beta)]}{\phi[d_{2,0}(1/\beta)]}\Phi[d_{2,0}(c_4/\beta)], \end{cases} \tag{32}$$

where $d_{1,0}(\cdot)$ and $d_{2,0}(\cdot)$ are defined in (31) with $t = 0$. The continuity of $H(\beta)$ follows immediately.

For each of the three cases (Cases A1, A2, and A3), $H(\beta)$ is a continuous function for $\beta > 0$, and moreover, $\xi_T x^*(\beta\xi_T)$ tends to 0 and ∞ respectively as β goes to ∞ and 0. Further, noticing the monotonicity of the function x^* (see Proposition 3.4), we get $\lim_{\beta\to\infty} H(\beta) = 0$ and $\lim_{\beta\to 0^+} H(\beta) = \infty$, and thus, there must be a constant $\beta > 0$ satisfying $\mathbb{E}[\xi_T Z^*_{\beta^*}] = x_0$. □

Remark 3.6. *The proof of Proposition 3.5 also implies that* $H(\beta) \equiv \mathbb{E}[\xi_T Z^*_\beta]$ *is non-decreasing as a function of* β *over the interval* $(0, \infty)$*. We solve for* β^* *numerically, and the observed monotonicity of* $H(\beta)$ *is a useful property in the root-finding procedure.*

4. Optimal Trading Strategy

In this section, we explore the optimal trading strategies π^* and $\widehat{\pi}$ based on the results obtained in the previous sections. We shall follow the martingale approach as outlined in the beginning of section 2.3, which entails computing both Y^*_t (resp. $\widehat{Y}_t$) and θ^*_t (resp. $\widehat{\theta}_t$) defined in equations (15) and (16) for the defautable policy (resp. full protected policy) and eventually obtaining the optimal trading strategy π^*_t (resp. $\widehat{\pi}_t$) via equation (17).

Hereafter, we use β^* and $\widehat{\nu}$ to denote two constants that satisfy $\mathbb{E}[\xi_T x^*(\beta^*\xi_T)] = x_0$ and $\mathbb{E}[\xi_T \widehat{x}(\widehat{\nu}\xi_T)] = x_0$ with the existence guaranteed by Proposition 3.5.

4.1. Optimal trading strategy for defaultable participating contract

The derivation of the optimal solution and portfolio value relies on the sequence of propositions and lemmas that we established in Sections 2 and 3. By part (a) of Proposition 3.4, the function $x^*(\cdot)$ defined there solves problem (22), and thus, by Lemma 3.1 and 3.2, $Z^* \equiv Z^*_{\beta^*} = x^*(\beta^*\xi_T)$ solves problem (13). Further, by Proposition 2.1, $\pi^* = \{\pi^*_t, 0 \le t \le T\}$ solves problem (11) with an optimal portfolio value at time t given by $X^{\pi^*}_t = Y^*_t, t \in [0, T]$ where $\pi^*_t = \sigma^{-1}\xi_t^{-1}\theta^*_t + \sigma^{-1}\zeta Y^*_t$ and $Y^*_t := \xi_t^{-1}\mathbb{E}[\xi_T Z^* | \mathcal{F}_t]$. The following proposition summarizes our results for the defaultable participating contract.

Proposition 4.1. *For the defaultable participating contract, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as follows with* β^* *satisfying* $\mathbb{E}[\xi_T X^*_T(\beta^*)] = x_0$*:*

Case A1. *If $1-\alpha>\gamma$, we define $\widetilde{z}=\frac{L_T^g}{1-\gamma}$ and $k=\gamma(\widetilde{z}-L_T^g)^{\gamma-1}$. Then, the optimal portfolio value at time t, $0\le t<T$, is given by*

$$\begin{cases} X_t^*(\beta^*)=\mathrm{e}^{-r(T-t)}(A_1+A_2+A_3+A_4+A_5), \\ \quad A_1=\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\left(\Phi[d_{2,t}(k/\beta^*)]-\Phi\left[d_{2,t}\left(m/\beta^*\right)\right]\right), \\ \quad A_2=L_T^g\left(\Phi[d_{1,t}(k/\beta^*)]-\Phi\left[d_{1,t}\left(m/\beta^*\right)\right]\right), \\ \quad A_3=\frac{L_T^g}{\alpha}\left(\Phi\left[d_{1,t}\left(m/\beta^*\right)\right]-\Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right), \\ \quad A_4=(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\Phi\left[d_{2,t}\left((1-\delta\alpha)m/\beta^*\right)\right], \\ \quad A_5=\frac{L_T^g(1-\delta)}{1-\delta\alpha}\Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]. \end{cases} \tag{33}$$

π_t^* given below is an optimal amount to invest in the risky asset at time t, for $0\le t<T$.

$$\begin{cases} \pi_t^*(\beta^*)=\frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(a_1+a_2+a_3+a_4+a_5), \\ \quad a_1=\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}K(k/\beta^*)-\left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}}K\left(m/\beta^*\right), \\ \quad a_2=L_T^g\left(\phi[d_{1,t}(k/\beta^*)]-\phi\left[d_{1,t}\left(m/\beta^*\right)\right]\right), \\ \quad a_3=\frac{L_T^g}{\alpha}\left(\phi\left[d_{1,t}\left(m/\beta^*\right)\right]-\phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right), \\ \quad a_4=(1-\delta\alpha)^{-1}\left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}}K\left[(1-\delta\alpha)m/\beta^*\right], \\ \quad a_5=\frac{L_T^g(1-\delta)}{1-\delta\alpha}\phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]. \end{cases} \tag{34}$$

Finally, the optimal terminal portfolio value is

$$\begin{aligned} X_T^*(\beta^*)=&\left[\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}}+L_T^g\right]\mathbf{1}_{\{m/\beta^*<\xi_T\le k/\beta^*\}}+\frac{L_T^g}{\alpha}\mathbf{1}_{\{(1-\delta\alpha)m/\beta^*\le\xi_T\le m/\beta^*\}} \\ &+\left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}}+\frac{(1-\delta)L_T^g}{1-\delta\alpha}\right]\mathbf{1}_{\{\xi_T<(1-\delta\alpha)m/\beta^*\}}. \end{aligned} \tag{35}$$

Case A2. *If* $(1-\delta\alpha)\gamma > 1-\alpha$, *we define* $\tilde{z} = \frac{(1-\delta)L_T^g}{(1-\delta\alpha)(1-\gamma)}$ *and* $k = \gamma(1-\delta\alpha)[(1-\delta\alpha)\tilde{z}-(1-\delta)L_T^g]^{\gamma-1}$. *Then, the optimal portfolio value at time* t, $0 \le t < T$, *is*

$$\begin{cases} X_t^*(\beta^*) = e^{-r(T-t)}(B_1 + B_2), \\ \quad B_1 = (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\dfrac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} \dfrac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\Phi[d_{2,t}(k/\beta^*)], \\ \quad B_2 = \dfrac{L_T^g(1-\delta)}{1-\delta\alpha}\Phi[d_{1,t}(k/\beta^*)]. \end{cases} \tag{36}$$

π_t^* *given below is an optimal amount to invest in the risky asset at time* t, *for* $0 \le t < T$.

$$\begin{cases} \pi_t^*(\beta^*) = \dfrac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(b_1 + b_2), \\ \quad b_1 = (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\dfrac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} K(k/\beta^*), \\ \quad b_2 = \dfrac{L_T^g(1-\delta)}{1-\delta\alpha}\phi[d_{1,t}(k/\beta^*)]. \end{cases} \tag{37}$$

Finally, the optimal terminal portfolio value is

$$X_T^*(\beta^*) = \left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1-\delta\alpha}\right] \mathbf{1}_{\{\xi_T < k/\beta^*\}}. \tag{38}$$

Case A3. *If* $\gamma \ge 1-\alpha \ge (1-\delta\alpha)\gamma$, *we define* $\tilde{z} = \frac{L_T^g}{\alpha}$ *and* $k = (1-\alpha)^{\gamma}(\tilde{z})^{\gamma-1}$. *Then, the optimal portfolio value at time* t, $0 \le t < T$, *is*

$$\begin{cases} X_t^*(\beta^*) = e^{-r(T-t)}(C_1 + C_2 + C_3), \\ \quad C_1 = (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\dfrac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} \dfrac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\Phi\left[d_{2,t}\left((1-\delta\alpha)m/\beta^*\right)\right], \\ \quad C_2 = \dfrac{L_T^g(1-\delta)}{1-\delta\alpha}\Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right], \\ \quad C_3 = \dfrac{L_T^g}{\alpha}\left(\Phi[d_{1,t}(k/\beta^*)] - \Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right). \end{cases} \tag{39}$$

π_t^* *given below is an optimal amount to invest in the risky asset at time* t, *for* $0 \le t < T$.

$$\begin{cases} \pi_t^*(\beta^*) = \dfrac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(c_1 + c_2 + c_3), \\ \quad c_1 = (1-\delta\alpha)^{-1} \left(\dfrac{m}{\gamma}\right)^{\frac{1}{\gamma-1}} K\left[(1-\delta\alpha)m/\beta^*\right], \\ \quad c_2 = \dfrac{L_T^g(1-\delta)}{1-\delta\alpha}\phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right], \\ \quad c_3 = \dfrac{L_T^g}{\alpha}\left(\phi[d_{1,t}(k/\beta^*)] - \phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right). \end{cases} \tag{40}$$

Finally, the optimal terminal portfolio value is

$$X_T^*(\beta^*) = \left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1-\delta\alpha}\right]\mathbf{1}_{\{\xi_T<(1-\delta\alpha)m/\beta^*\}} + \frac{L_T^g}{\alpha}\mathbf{1}_{\{(1-\delta\alpha)m/\beta^*\le\xi_T\le k/\beta^*\}}. \tag{41}$$

Proof. **Step 1.** Obtain the terminal portfolio value $X_T^*(\beta^*) := X_T^{\pi^*}(\beta^*) = Z_{\beta^*}^* \equiv x^*(\beta^*\xi_T)$ and the portfolio value at t, i.e.

$$X_t^*(\beta^*) := X_t^{\pi^*}(\beta^*) \equiv Y_t^* = \xi_t^{-1}\mathbb{E}[\xi_T Z_{\beta^*}^*|\mathcal{F}_t] \equiv \xi_t^{-1}\mathbb{E}[\xi_T x^*(\beta^*\xi_T)|\mathcal{F}_t].$$

In this step, the formulas given in Appendix B are useful. The obtained $X_t^*(\beta^*)$ depends on t and ξ_t, and thus we can write $X_t^*(\beta^*) = q(t,\xi_t)$, where q is a C^2 function as one can see from equations (33), (36) and (39).

Step 2. We note that $\{\xi_t Y_t^*, \forall 0 \le t \le T\}$ is a martingale with $\xi_0 Y_0^* = x_0$ so that it has a zero drift. Thus, from equation (8), we obtain $d\xi_t = -r\xi_t dt - \zeta\xi_t dW_t$, and further apply Itô's formula to get the diffusion term of $\xi_t Y_t^*$ as follows

$$\theta_t^* = -\zeta\xi_t\left(Y_t^* + \xi_t\frac{\partial q(t,\xi_t)}{\partial\xi_t}\right).$$

Step 3. Apply equation (17) to obtain the optimal trading strategy by the formula

$$\pi_t^* = \sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^* = -\frac{\zeta\xi_t}{\sigma}\frac{\partial q(t,\xi_t)}{\partial\xi_t}.$$

The specific implementation of the above three-step procedure for results in case A1 is demonstrated in Appendix C, and the results for the other two cases can be obtained similarly. □

4.2. Optimal trading strategy for fully protected participating contract

Proposition 4.2. *For the fully protected participating contract, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as below with $\widehat{\nu}$ satisfying $E[\xi_T X_T^*(\widehat{\nu})] = x_0$:*

Case B1. *If $\lambda > \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, let $\tilde{z}$ be the unique solution of equation* (29) *over the interval $(L_T^g, \frac{L_T^g}{\alpha})$ and let $k = \gamma(\widetilde{z} - L_T^g)^{\gamma-1}$ as in Case B1, part (b) of Proposition 3.4. Then, the optimal portfolio value at time t, $0 \le t < T$, is $X_t^*(\widehat{\nu})$ given by* (33)*, the optimal trading strategy is $\pi^*(\widehat{\nu})$ given by* (34)*, and the optimal terminal portfolio value is $X_T^*(\widehat{\nu})$ given by* (35)*.*

Case B2. *If* $\lambda < \frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$*, then let* $\tilde{z}$ *be the unique solution to equation* (30) *over the interval* $(\frac{L_T^g}{\alpha},\infty)$*, and define* $k=\gamma(1-\delta\alpha)[(1-\delta\alpha)\tilde{z}-(1-\delta)L_T^g]^{\gamma-1}$ *as in Case B2, part (b) of Proposition 3.4. Then the optimal portfolio value at time* t*,* $0 \le t < T$ *is* $X_t^*(\widehat{\nu})$ *given by* (36)*, the optimal trading strategy is* $\pi^*(\widehat{\nu})$ *given by* (37)*, and the optimal terminal portfolio value is* $X_T^*(\widehat{\nu})$ *given by* (38)*.*

Case B3. *If* $\frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1} \le \lambda \le \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$*, define*

$$\tilde{z}=\frac{L_T^g}{\alpha} \ \text{ and } \ k=\alpha\left[\left(\frac{1-\alpha}{\alpha}\right)^{\gamma}+\lambda\right](L_T^g)^{\gamma-1}.$$

Then the optimal portfolio value at time t*,* $0 \le t < T$*, is* $X_t^*(\widehat{\nu})$ *given by* (39)*, the optimal trading strategy is* $\pi^*(\widehat{\nu})$ *given by* (40)*, and the optimal terminal portfolio value is* $X_T^*(\widehat{\nu})$ *given by* (41)*.*

Proof. For each of Case B1, B2, and B3, the results can be derived following a three-step procedure in a similar way as in Proposition 4.1. □

Remark 4.3. *For both the defaultable and protected policies, both* $X_T^*(\beta^*)$ *and* $X_T^*(\widehat{\nu})$ *are sums of indicator functions, which are non-negative. The non-negativity of both* $X_t^*(\beta^*)$ *and* $X_t^*(\widehat{\nu})$ *follows from their derivation as in Proposition 2.1. Meanwhile,* π_t^* *and* $\widehat{\pi}_t$ *are actually non-negative as well; see Appendix D for a more detailed explanation.*

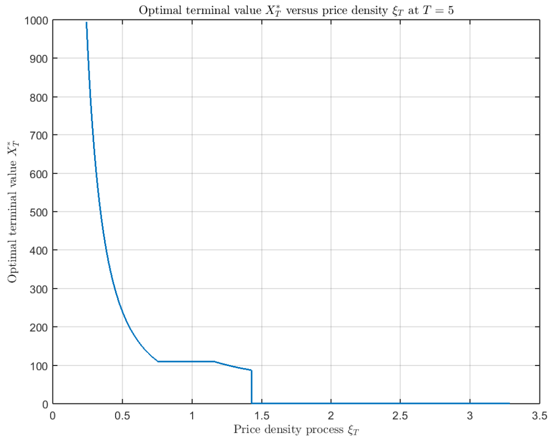
5. Numerical Examples

In this section, we numerically implement the results obtained in the Propositions 4.1 and 4.2 for illustration. For notational convenience, we suppress the argument β^* and write $X_t^*(\beta^*)$ and $\pi_t^*(\beta^*)$ as simply X_t^* and π_t^* respectively. We consider parameters chosen as follows:

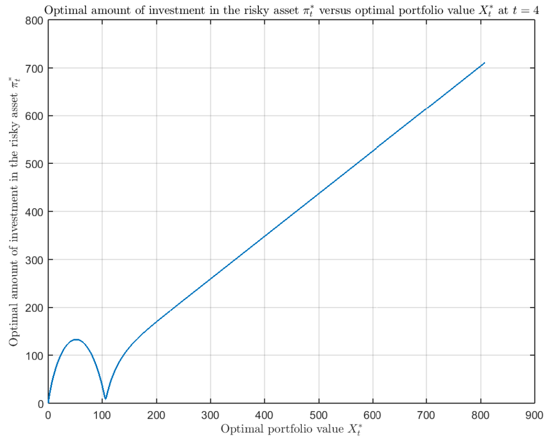
x_0	T	r	g	μ	σ
100	5	0.03	0.0175	0.07	0.3

Table 1: Parameter Setting for Numerical Illustration

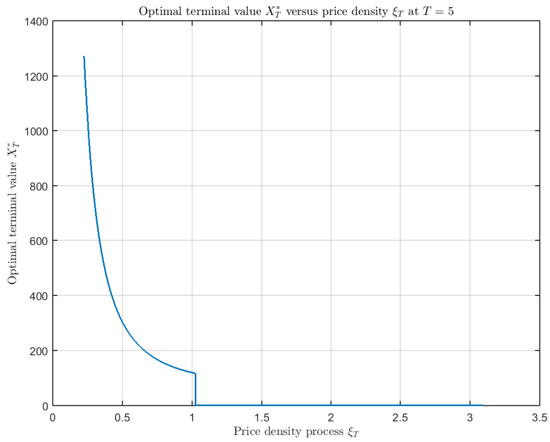
We select $T = 5$ instead of a longer term since constant parameters are assumed. Because the condition for each case in Proposition 4.1 and 4.2 varies, we conduct the numerical illustration based on different choices of α, δ, γ, and λ that result in the different cases. The results are given in Figure 3.



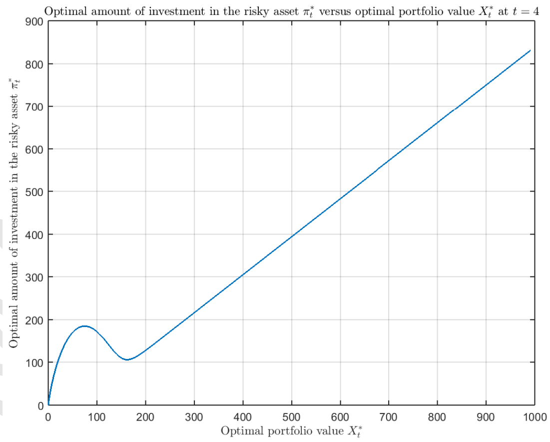
(a) Case A1: $\alpha = 0.4$



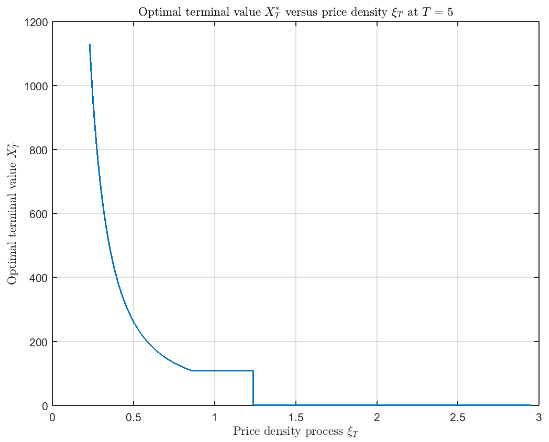
(b) Case A1: $\alpha = 0.4$



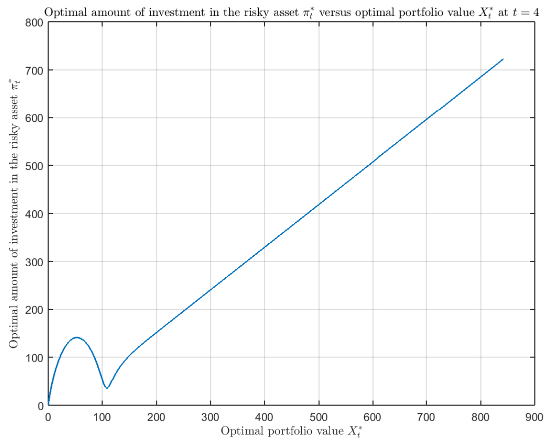
(c) Case A2: $\alpha = 0.9$



(d) Case A2: $\alpha = 0.9$



(e) Case A3: $\alpha = 0.75$



(f) Case A3: $\alpha = 0.75$

Figure 3: Defaultable participating contract with $\gamma = 0.5$, $\delta = 0.8727$. Left panel: optimal terminal value versus price density process. Right panel: Optimal amount of investment in the risky asset versus optimal asset value.

The left panel of Figure 3 shows the optimal terminal value X_T^* versus the price density process ξ_T. Recall from (35), (38) and (41) that X_T^* is the summation of indicator functions and when $\xi_T > k/\beta^*$, $X_T^* = 0$. The figures in the left panel reveal the value of k/β^* for each case. Meanwhile, as expected, we can obtain the solutions β^* to $\mathbb{E}[\xi_T X_T^*] = x_0$ for Cases A1, A2, and A3, which are 0.053, 0.0296 and 0.0387 respectively.

The right panel of Figure 3 illustrates the optimal investment amount in the risky asset π_t^* versus the total optimal portfolio value X_t^* at time $t = 4$, i.e., one year before maturity. As revealed by the figures, π_t^* versus the optimal value X_t for Cases A1, A2, and A3 (Figures 3b, 3b and 3b) exhibits a "peak-and-valley" pattern with distinct kink points. X_t^* is non-negative, coinciding with our theoretical finding in Proposition 4.1; (see Remark 4.3). When the optimal value X_t^* is close to zero, the optimal investment amount in the risky asset stays close to zero as well. When X_t^* is large enough, at least larger than the value of the second turning point shown in the figures, the optimal investment amount in the risky asset π_t^* increases with X_t^*.

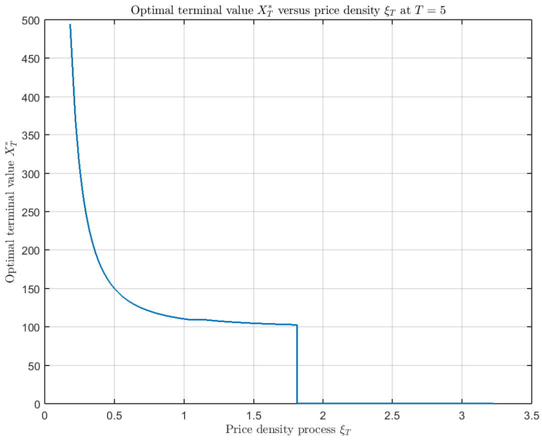
Figure 4 provides numerical illustrations for the protected policy. From the figures, we can see that different cases exhibit similar patterns with slight differences depending on the choices of parameters.

5.1. Comparison with CPPI strategy

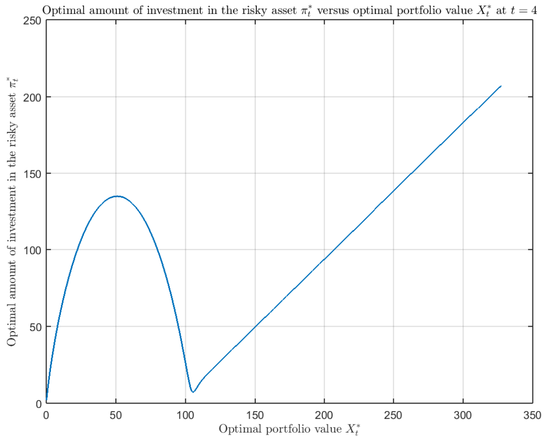
Bernard and Le Courtois [2] considered the Constant Proportion Portfolio Insurance (CPPI) strategy for asset management with participating contracts. In theory as well as in practice, CPPI has shown its advantage in that the strategy not only guarantees a minimum level of wealth over a pre-specified time horizon, but also allows potential upward return. In this respect, it is well-designed because it protects investors from downside risk and provides an opportunity to earn excess return when the market performs well. At each time, the discounted guarantee is called the floor, and the investment in the risky asset is proportional to the cushion value, defined as the portfolio value less the floor. The proportional factor is called the multiplier of CPPI. See Chapter 9 in Prigent [20] for more technical details regarding the CPPI strategy.

Under a geometric Brownian motion model for the risky asset, the value process of a CPPI portfolio, as shown in Prigent [20], is as follows:

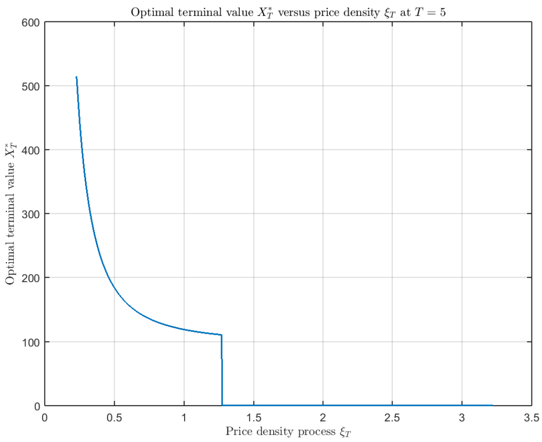
$$V_t^{CPPI}(m, S_t) = F_0 e^{rt} + C_0 \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - \frac{m^2\sigma^2}{2}\right] t\right\} \left(\frac{S_t}{S_0}\right)^m, \ \ 0 \le t \le T,$$



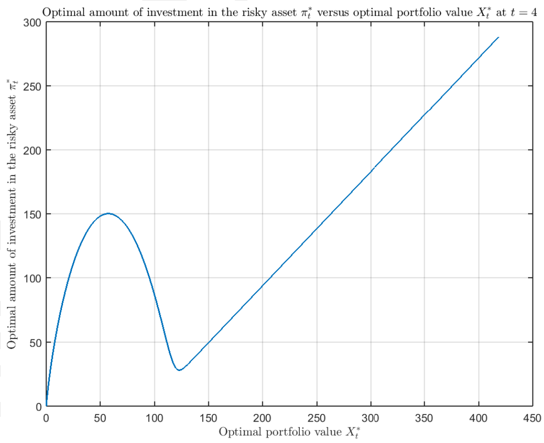
(a) Case A: $\lambda = 2.25$



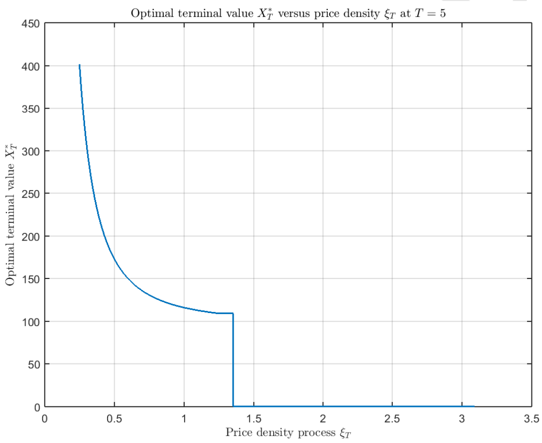
(b) Case A: $\lambda = 2.25$



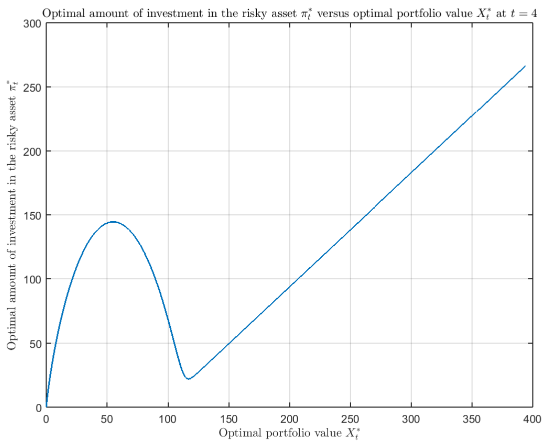
(c) Case B: $\lambda = 1.1$



(d) Case B: $\lambda = 1.1$



(e) Case C: $\lambda = 1.3$



(f) Case C: $\lambda = 1.3$

Figure 4: Protected policy with $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$. Left panel: optimal terminal value versus price density process. Right panel: Optimal amount of investment in the risky asset versus optimal asset value.

where, m is the multiplier for CPPI, C_0 and F_0 are the initial cushion and initial floor, respectively. The value process $V_t := V_t^{CPPI}(m, S_t)$, floor process F_t and cushion process C_t are related by $V_t = F_t + C_t$ for $t \in [0, T]$.

For comparison, we set the guarantee floor as $F_T = F_0 e^{rT} = L_0 e^{gT}$. Since S_t is assumed to follow a geometric Brownian motion, it is easy to verify that

$$\begin{cases} \mathbb{E}\left[V_t^{CPPI}(m, S_t)\right] = F_0 e^{rt} + C_0 \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - \frac{m^2\sigma^2}{2}\right] t\right\} \exp(\mu_{\text{cppi}} + \frac{1}{2}\sigma_{\text{cppi}}^2) \\ \sqrt{Var\left[V_t^{CPPI}(m, S_t)\right]} = C_0 \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - \frac{m^2\sigma^2}{2}\right] t\right\} \sqrt{\left(e^{\sigma_{\text{cppi}}^2} - 1\right) e^{2\mu_{\text{cppi}} + \sigma_{\text{cppi}}^2}} \end{cases}$$

where $\mu_{\text{cppi}} = (\mu - \frac{1}{2}\sigma^2)mT$ and $\sigma_{\text{cppi}} = \sigma m \sqrt{T}$.

With $X_t = V_t^{CPPI}(m, S_t), \forall 0 \le t \le T$, we have no analytical formula for $\mathbb{E}\left[U(\Psi(X_T))\right]$ and $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$. We rely on simulation to estimate these values.

The parameters are specified in Table 2. Similarly, $T = 5$ is selected instead of a longer term since we assume constant parameters. $\lambda = 2.25$ is set following the paper by He and Kou [14]. The number of simulations is set to be $N = 10000$. Additionally, we choose $\sigma = 0.1$, $\sigma = 0.3$ and $\sigma = 0.5$ to represent markets with different volatility levels. As for the CPPI strategy, $m = 0.5$, $m = 1$ and $m = 1.5$ are selected to represent a conservative strategy, moderate strategy and aggressive strategy, respectively. The parameters used in the examples are summarized in Table 2.

x_0	T	r	g	μ	α	δ	γ	λ	N
100	5	0.03	0.0175	0.07	0.9	0.8727	0.5	2.25	10000

Table 2: Parameter Setting for Comparison

The numerical results are shown in Table 3. As would be expected (since we are looking at in-sample results), the expected utility from the optimal strategy in both the defaultable and protected policies is always greater than that from the standard CPPI strategy across all the three levels of volatility.

Secondly, notice that in the stable market, i.e. $\sigma = 0.1$, the expected utility from the standard CPPI strategy increases with m. However, when $\sigma = 0.3$ and $\sigma = 0.5$, as m increases, the insurance company will be less satisfied, i.e. the expected utility decreases. Therefore, adopting more aggressive CPPI strategy, i.e. a large m, will result in less satisfaction in the presence of a large σ.

Volatility	Strategy		$\mathbb{E}[U(\Psi(X_T))]$	$\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$
$\sigma = 0.1$	CPPI	$m = 0.5$	3.5820	3.5820
		$m = 1$	3.6424	3.6424
		$m = 1.5$	3.7016	3.7016
	DP		6.5364	×
	PP		×	4.5237
$\sigma = 0.3$	CPPI	$m = 0.5$	3.5701	3.5701
		$m = 1$	3.518	3.518
		$m = 1.5$	3.3388	3.3388
	DP		3.9592	×
	PP		×	3.6105
$\sigma = 0.5$	CPPI	$m = 0.5$	3.4827	3.4827
		$m = 1$	3.1642	3.1642
		$m = 1.5$	2.6563	2.6563
	DP		3.7072	×
	PP		×	3.5685

Table 3: Comparison Statistics with $x_0 e^{rT} = 116.1834$ and $L_0 e^{gT} = 98.2298$: DP(resp. PP) stands for our strategy in defaultable policy(resp. protected policy); "×" stands for "not applicable".

Thirdly, note that when σ changes from 0.1 to 0.3, our strategy both in the defaultable and protected policies results in the decrease of expected utility by roughly 2.57 and 0.91, respectively. As well, the CPPI strategy leads to decrease of expected utility by 0.012, 0.12, and 0.36 for $m = 0.5$, $m = 1$ and $m = 1.5$, respectively. But when σ changes from 0.3 to 0.5, the expected utility for the insurance using our strategy in the two policies decreases by approximately 0.25 and 0.05, respectively. For the CPPI strategy, the expected utility decreases by 0.08, 0.35, and 0.68 for $m = 0.5$, $m = 1$, and $m = 1.5$, respectively.

In short, theoretically it is possible that the portfolio value may fall below the guarantee level, resulting in nothing for the insurance company selling defaultable participating contracts and a negative payoff for the one selling protected policies, compared with the CPPI strategy which always leads to an asset value above the guarantee level. When employing CPPI in practice, one cannot continuously rebalance the portfolio. Consequently, it is possible that the portfolio value may fall below the guarantee level when using a discretized CPPI strategy. The difference between the optimal utility and the CPPI utility is more pronounced when $\sigma \in (0.1, 0.3)$ than when $\sigma \in (0.3, 0.5)$.

5.2. On comparison with OBPI strategy

Another approach considered by Bernard and Le Courtois [2] is the Option Based Portfolio Insurance (OBPI) strategy. Its goal is to guarantee the investor a terminal portfolio value never below a certain level for a given time horizon. In theory, this is a strategy constructed via purchasing European put options and the corresponding underlying assets, or buying bonds and call options. However, in practice, the strategy is often impossible to implement because there are in general no available options for a given maturity. One possibility is to use Equity Default Swaps (EDSs) which have longer maturities than standard options. This is examined in Bernard and Le Courtois [2] and Bernard et al. [3].

EDSs are created for the similar reason as CDSs, which protect against severe events on bonds. The investor in EDSs pays a fee periodically, typically semi-annually. When an equity falls by $100d\%$ of its initial value then the severe event occurs and the investor will be given a rebate. A common choice of barrier level (i.e. $(1 - 100d\%)$) is 70%. For the rebate setting, in Bernard et al. [3], they chose 50% of initial value as the rebate, i.e. $50\% \times S_0$, while Bernard and Le Courtois [2] use 50% of the dropped value, i.e. $50\% \times 100d\% \times S_0$. We follow the latter reference. The maturity of EDSs varies, and is typically set equal to 5 years.

Note that the EDSs terminate at the first time τ such that $S_\tau = (1 - d)S_0$. If the underlying does not touch the barrier level $(1 - d)S_0$, the investor in EDSs ends up with a zero payoff at maturity. The density $g_\tau(t)$ of the first-hitting time τ is given by

$$g_\tau(t) = \frac{|\ln(1-d)|}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(|\ln(1-d)| - |r - 0.5\sigma^2|t)^2}{2\sigma^2 t}\right)$$

This is an Inverse Gaussian distribution with $\lambda = \left[\frac{\ln(1-d)}{\sigma}\right]^2$ and $\mu = \left|\frac{\ln(1-d)}{r-0.5\sigma^2}\right|$, denoted as $IG(\lambda, \mu)$. Here we set $S_0 = x_0 = 100$. The rebate is set to $50\% \times 100d\% \times S_0 = 0.5dS_0$, thus the expected discounted payoff is

$$E(0.5dS_0 e^{-r\tau}\mathbf{1}_{\tau<T}) = 0.5dS_0 \int_0^T e^{-r\tau} g_\tau(t)dt = 0.5dS_0 \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2 r}{\lambda}}\right)\right] \int_0^T g_{\tau_{\text{eds}}}(t)dt$$

where τ_{eds} follows $IG(\lambda_{\text{eds}}, \mu_{\text{eds}})$ with $\lambda_{\text{eds}} = \lambda$ and $\mu_{\text{eds}} = \frac{\mu}{\sqrt{1-2\mu^2 t/\lambda}}$. Therefore, we can solve it explicitly.

The portfolio consists of n shares of stock S and EDSs, i.e. $x_0 = n \times [S_0 + \mathbb{E}(0.5dS_0e^{-r\tau}\mathbf{1}_{\tau<T})]$. Following the parameters specified above and in Table 2, we choose $d = 1 - \frac{L_0e^{gT}}{S_0} = 0.0177$, $d = 0.3$ and $d = 0.5$ representing the barrier level to be the guarantee liability, 70% of the initial equity value and 50% of the initial equity value, respectively. We assume that when the stock price hits the barrier level, all the money, including the rebate and the amount of money from the sale of stock, will be invested in the risk-free asset.

Volatility	Strategy		$\mathbb{E}[U(\Psi(X_T))]$	$\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$
$\sigma = 0.1$	EDS	$d = 0.0177$	3.7407	3.7407
		$d = 0.3$	3.7337	3.2129
		$d = 0.5$	3.9278	3.5685
	DP		6.5364	×
	PP		×	4.5237
$\sigma = 0.3$	EDS	$d = 0.0177$	3.5967	3.5967
		$d = 0.3$	2.1714	-2.3837
		$d = 0.5$	2.6529	-2.2039
	DP		3.9592	×
	PP		×	3.6105
$\sigma = 0.5$	EDS	$d = 0.0177$	3.5691	3.5691
		$d = 0.3$	1.2512	-5.3787
		$d = 0.5$	1.8190	-6.3325
	DP		3.7072	×
	PP		×	3.5685

Table 4: Comparison Statistics with $x_0e^{rT} = 116.1834$ and $L_0e^{gT} = 98.2298$: DP (resp. PP) stands for our strategy in defaultable policy (resp. protected policy); "×" is short for "not applicable".

The numerical results are shown in Table 4. The OBPI strategy has a lower utility than the optimal strategy (as is to be expected, since we are looking at in-sample results), with the difference being larger at a high volatility level.

In addition, given a volatility level, when the barrier level is set close to the initial equity value, which means it is easy to reach the barrier level, the portfolio consists of only the risk-free asset after touching the barrier level. In this case, the portfolio evolves like the risk-free asset. Although the EDS protects the portfolio from falling below the barrier level, it does not allow potential upward return once it touches the barrier level. The utility in this case is close to that from simply investing in the risk-free asset. However, when the barrier level is set far below the initial equity value, the premium of the EDS is high. The investment shares in both the equity and EDSs are small due to the budget. In the future, if the price of the stock declines to the barrier level, the rebate will be returned to the insurer, otherwise the insurance company will get no payoff from the EDSs. In other words, the expected utility is not very high mainly due to the small value of the terminal portfolio resulting from the small shares in both the stock and EDSs, although the upside return of the equity might be large.

The (in-sample) out-performance of the optimal strategy in terms of utility confirms the analytical results given earlier. Furthermore, when the volatility level is changing but still stays high, the optimal strategy performs better due to the small change of the expected utility level. As we will see in the next section, when σ is high, the expected utility from the optimal strategy is close to that with a bounded constraint on the control. In other words, when σ is high, the optimal strategy dynamically chooses not to invest too much money in the risky asset, which is different from the standard CPPI and OBPI strategy where the multiplier m is set to be constant at the beginning.

6. Constrained Optimization Problem with Bounded Control

For both the defaultable and protected policies, our numerical experiments (Figures 3 and 4) show that it is possible to have $\pi_t^* > X_t^*$ for some $t \in [0, T]$, i.e. the amount of money invested in the risky asset is greater than the total portfolio value at time t. In other words, the insurance company takes a leverage position in the risky asset by borrowing money. This increases expected utility and expected return, but produces a riskier portfolio and may violate investment policies. In this section, we consider the utility maximization problem with a constraint placing an upper bound on the control.

6.1. The formulation

We rewrite the dynamics for the portfolio given in (1) by introducing the portfolio weight in the risky asset η_t to obtain

$$dX_t^\eta = [r + \eta_t(\mu - r)]X_t^\eta dt + \sigma\eta_t X_t^\eta dW_t. \tag{42}$$

We consider the constraint set of the control $\Sigma := [0, \eta_{\max}]$ where we set $\eta_{\max} = 0.4$. We denote $\mathcal{C}$ as the set of $\mathbf{F}$-progressively measurable processes η such that $\eta_t \in \Sigma, \forall 0 \leq t \leq T$ a.s. Then the constrained optimization problems for the defaultable and protected policies can be written as

$$\sup_{\eta\in\mathcal{A}(x_0)\cap\mathcal{C}} \mathbb{E}[U(\Psi(X_T^\eta))] \text{ and } \sup_{\eta\in\mathcal{A}(x_0)\cap\mathcal{C}} \mathbb{E}[U(\widehat{\Psi}(X_T^\eta))], \tag{43}$$

where $\mathcal{A}(x_0)$ is given in (7).

Denote by $v(t,x)$ the optimal objective value of the problem, evaluated at time t given that $X_t^\pi = x$. It can be shown that the solution to the following HJB equation coincides with $v(t,x)$ (see Chapter 3 in Pham [19]):

$$\begin{cases} v_t + xv_x r + \sup_{\eta_t\in\Sigma}\{xv_x\eta_t(\mu - r) + \frac{1}{2}x^2\sigma^2\eta_t^2 v_{xx}\} = 0, \\ v(T,x) = U(\Psi(x)). \end{cases} \tag{44}$$

It can be proved that the optimal objective value function $v(t,x)$ is the viscosity solution to the above HJB equation by Pham (2009, [19], Theorems 4.3.1 and 4.3.2, pp. 68-69). The uniqueness of the viscosity solution to the above HJB equation can be justified by Fleming and Sonner (2006, Section V.9, pp. 222-224) [10]. Similarly, the constrained optimization problem for the protected contract can be formulated and we will have the same partial differential equation as the above with $\Psi(\cdot)$ replaced by $\widehat{\Psi}(\cdot)$ in the boundary condition.

6.2. Optimal value under constrained optimization

To solve the HJB equation numerically, we use the scheme proposed by Forsyth and Labahn [11]. Using the parameter values in Table 1, we solve the constrained optimization problems varying the choices of α, δ, γ and λ for comparison.

We define a grid by discretizing both the state space and time. Following the parameters in Table 2, we carry out a numerical experiment to find out the optimal value $v(0, x_0)$ for three distinct values of σ, as shown in Table 5. x-nodes refers to the discretized state space, while time steps is the total number discretized time steps. We conduct our numerical experiment by fully implicit scheme and Crank-Nicolson scheme. The results for both schemes are similar, thus we only report the result from the fully implicit method with constant time-step.

Firstly, as the numbers of x-nodes and time steps increase, the number of iterations taken until convergence increases. Here, we discretize the control space and obtain the π-nodes. We use a linear search method to determine the optimal control value (see Sections 4 and Section 7 in Wang and Forsyth [24]) because of the complexity of the form of the HJB equation and the Positive Coefficient condition. Note that we keep the number of π-nodes constant. Increasing the number of π-nodes and yields similar results.

Secondly, as expected, the optimal value under the constrained optimization problem obtained via the numerical PDE method is always smaller than the optimal value for the unconstrained optimization problem using simulation given the analytical solution for optimal terminal wealth derived in the previous sections.

Thirdly, it is worth mentioning that the optimal value for the unconstrained optimization problem can also be obtained via the numerical PDE method. We have also carried out the numerical experiment by choosing $\eta_{\max}$ to be large enough and attain values very close to those derived from the analytical solution.

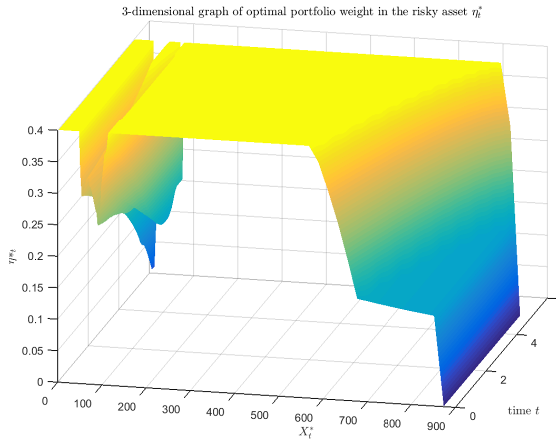
Finally, when σ increases, it seems that the difference of optimal values between the constrained and unconstrained problems becomes smaller. In other words, the portfolio evolves as if it is unconstrained. When σ is large, a small change of σ does not cause too much difference in the optimal value. Therefore, the strategy is less sensitive to σ in a volatile market, which agrees with our previous finding.

6.3. Portfolio weight under constrained optimization

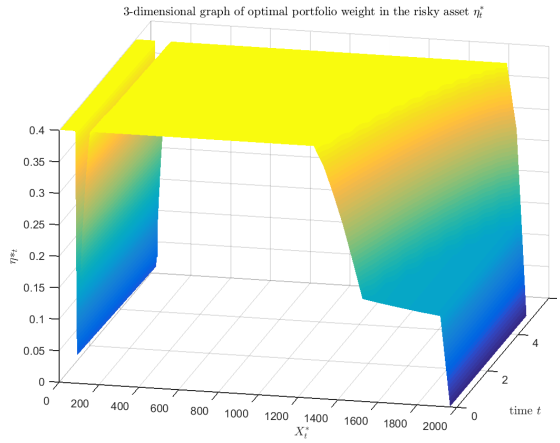
The results are shown in Figures 5 and 6. Figure 5 exhibits a three-dimensional graph of the optimal portfolio weight in the risky asset at time $t = 4$, one year before maturity of the contract over all possible portfolio values X_t^*. When the portfolio value is large enough, the exhibited patterns are similar, while they are slightly different when the portfolio values are roughly between 0 to 200, which is the area of our interest because the insurance company is endowed with 100 initially. Notice that the weight is capped at 0.4, which is different from the unconstrained problem in which there is no bound on the control.

			Defaultable Policy		Protected Policy	
number of x-nodes	time steps	number of π-nodes	iterations	$\mathbb{E}\left[U(\Psi(X_T))\right]$ constrained	iterations	$\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$ constrained
$\sigma = 0.1$, $x_0 = 100$, $\mathbb{E}\left[U(\Psi(X_T))\right] = 6.5342$, $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right] = 4.5237$ (unconstrained case)						
73	60	1000	178	3.781671	185	3.781658
145	120	1000	312	3.779824	346	3.779669
289	240	1000	542	3.775911	607	3.775515
577	480	1000	1004	3.773042	1087	3.77247
1153	960	1000	1975	3.771647	2042	3.770973
2305	1920	1000	3907	3.77092	3954	3.770191
4609	3840	1000	7769	3.770584	7782	3.769826
$\sigma = 0.3$, $x_0 = 100$, $\mathbb{E}\left[U(\Psi(X_T))\right] = 3.9592$, $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right] = 3.6105$ (unconstrained case)						
73	60	1000	190	3.604185	192	3.604098
145	120	1000	350	3.604694	373	3.60466
289	240	1000	584	3.605178	653	3.605159
577	480	1000	1052	3.605903	1122	3.605892
1153	960	1000	2037	3.606696	2056	3.60669
2305	1920	1000	4009	3.607194	3942	3.607191
4609	3840	1000	7906	3.607708	7714	3.607706
$\sigma = 0.5$, $x_0 = 100$, $\mathbb{E}\left[U(\Psi(X_T))\right] = 3.7072$, $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right] = 3.5685$ (unconstrained case)						
73	60	1000	194	3.55641	200	3.555774
145	120	1000	360	3.560061	379	3.560002
289	240	1000	619	3.56047	656	3.560446
577	480	1000	1094	3.560912	1109	3.5609
1153	960	1000	2105	3.561317	2035	3.561311
2305	1920	1000	4098	3.561609	3877	3.561605
4609	3840	1000	8001	3.561886	7708	3.561884

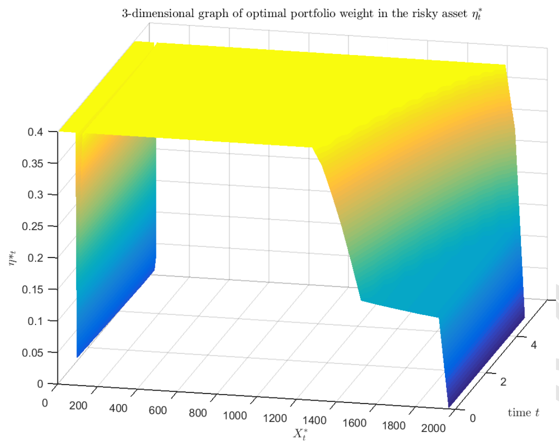
Table 5: Fully Implicit Method with a constant time steps for constrained optimization with a bounded control. The portfolio weight $\eta_t \in [0, 0.4], \forall t \in [0, T]$.



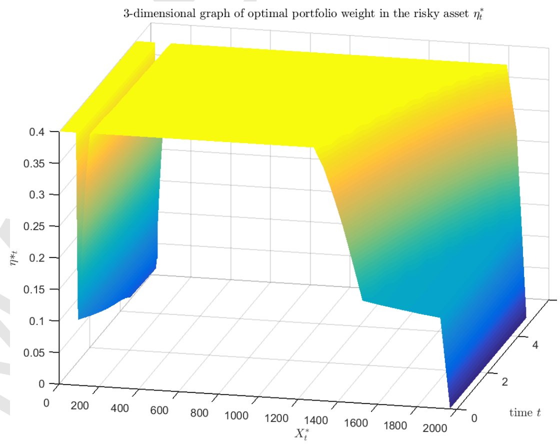
(a) Case A1: $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8727$



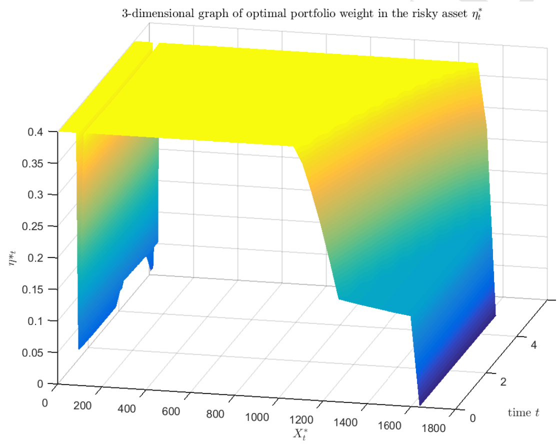
(b) Case B1: $\lambda = 2.25$,$\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$



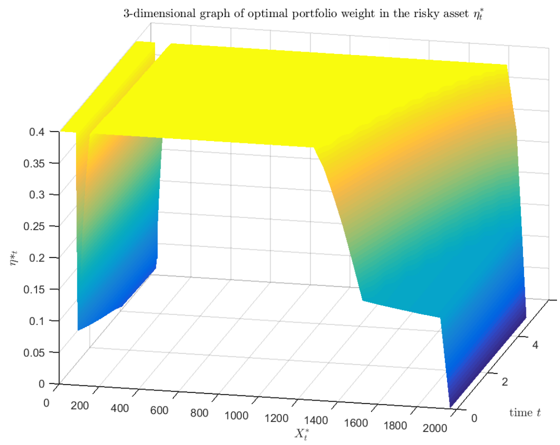
(c) Case A2: $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.8727$



(d) Case B2: $\lambda = 1.1$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$

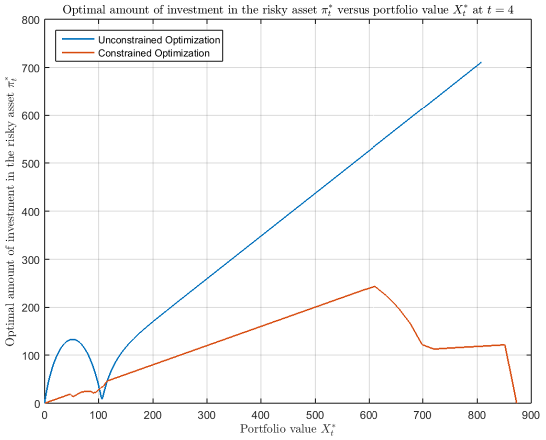


(e) Case A3: $\alpha = 0.75$, $\gamma = 0.5$, $\delta = 0.8727$

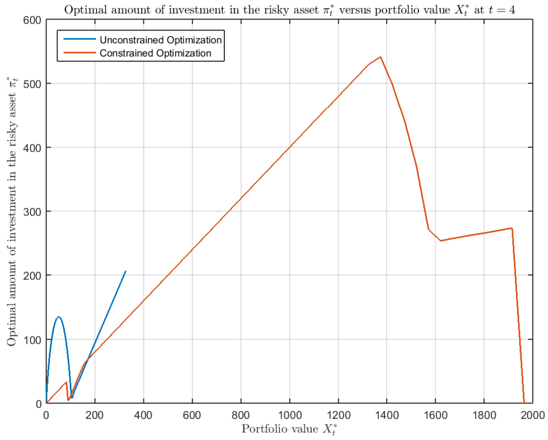


(f) Case B3: $\lambda = 1.3$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$

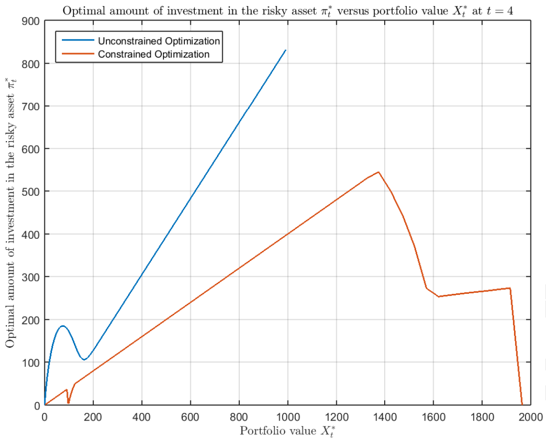
Figure 5: Optimal weight under constrained optimization. Left panel: Defaultable participating contract. Right panel: Protected participating contract.



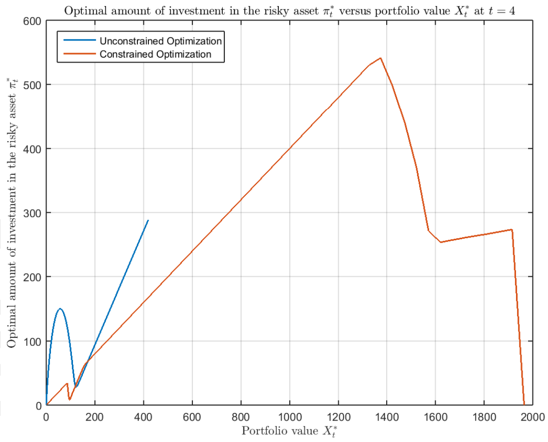
(a) Case A1: $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8727$



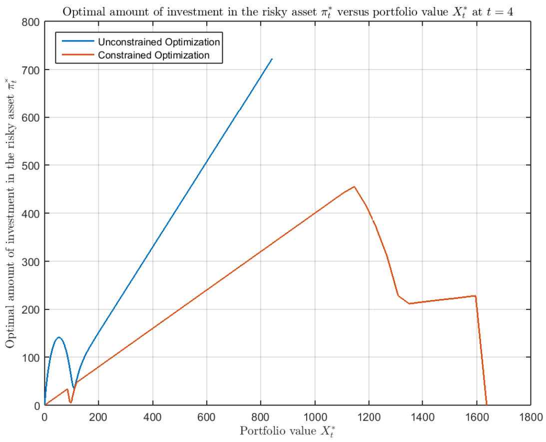
(b) Case B1: $\lambda = 2.25, \alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$



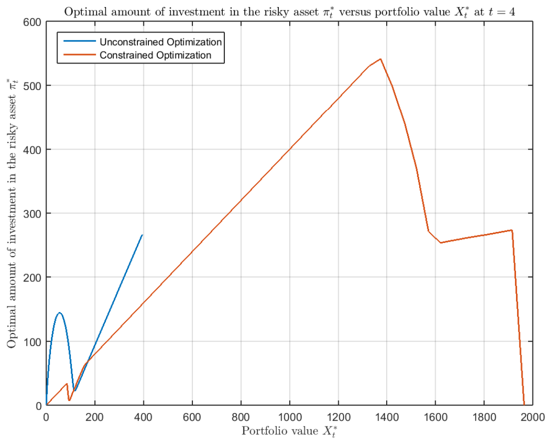
(c) Case A2: $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.8727$



(d) Case B2: $\lambda = 1.1$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$



(e) Case A3: $\alpha = 0.75$, $\gamma = 0.5$, $\delta = 0.8727$



(f) Case B3: $\lambda = 1.3$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$

Figure 6: Optimal amount of investment in the risky asset versus portfolio value value. Left panel: Defaultable participating contract. Right panel: Protected participating contract.

To better illustrate the difference from unconstrained optimization, Figure 6 shows the optimal amount invested in the risky asset at $t = 4$ for both the constrained and unconstrained problems. As shown in Figure 5, the graph exhibits a slight difference when the asset value is between 0 to 200. However, most of the figures show a "peak-and-valley" pattern and π_t^* increases with X_t^* when X_t^* is larger than the value of the second of the 'turning points'. It is worth mentioning that for the unconstrained optimization problem, the result is obtained using (34), (37) and (40) by simulation, therefore the range of the value X_t^* differs from the constrained optimization problem, in which we manually select the range of X_t^* while implementing numerical PDE method. Furthermore, as was pointed out previously by Barles et al. [1] and Forsyth and Labahn [11], the possible maximum value of X_t^* should be set to be large enough as to make the error incurred from the approximating boundary condition to be small in the area of our interest. The similarity of the patterns when the portfolio value is large in Figures 5 and 6 is due to our choice of approximating boundary condition, which is set to be independent of time.

7. Discussion

In this paper, we consider a portfolio selection problem for a utility maximizing insurance company selling participating contracts. Relying on the martingale approach and the pointwise optimization technique, we are able to obtain a closed-form solution. In the pointwise optimization procedure we adopt a concavification technique to transform the problem to a solvable one. With the optimal solution, we present numerical examples as well as comparisons with the standard CPPI and OBPI strategies. Finally, we consider a constrained version of the optimization problem with bounded control, obtain the solution by employing a numerical method, and compare the solutions of the constrained and unconstrained problems.

Appendix.

A. Lemmas Used for Proving Proposition 3.4

Since the functions we deal with are eventually concave, their concave envelopes can be found by calculating a single tangent line. This is formalized in the following lemma.

Lemma A.1. *Suppose $f : [0, \infty) \to [0, \infty)$ is continuous and satisfies:*

1. $f(0) = 0$.
2. f *is concave on* $[\tilde{z}, \infty)$, *with* $\tilde{z} > 0$.
3. $f(x) \leqslant kx$ *on* $[0, \tilde{z}]$, *with* $k = \frac{f(\tilde{z})}{\tilde{z}} > 0$.
4. $k \geqslant f'_+(\tilde{z})$.

Then the concave envelope of f is:

$$f^c(x) = \begin{cases} kx, & x \in [0, \tilde{z}), \\ f(x), & x \in [\tilde{z}, \infty). \end{cases} \tag{A.1}$$

Proof. By definition $f^c \geqslant f$. Let g be concave with $g \geqslant f$. Then $g \geqslant f^c$ on $\{0\} \cup [\tilde{z}, \infty)$, since $f^c = f$ there. Suppose $x \in (0, \tilde{z})$, i.e. $x = \lambda\tilde{z}$ for $\lambda \in (0, 1)$. By concavity of g:

$$g(x) = g(\lambda\tilde{z} + (1-\lambda)\cdot 0) \geqslant \lambda g(\tilde{z}) + (1-\lambda)g(0) \geqslant \lambda k\tilde{z} = kx = f^c(x).$$

It remains to show that f^c is concave. Let $x_0, x_1 \in [0, \infty)$ with $x_0 < x_1$ and $x_\lambda = \lambda x_0 + (1-\lambda)x_1$ with $\lambda \in (0, 1)$. The inequality $f^c(x_\lambda) \geqslant \lambda f^c(x_0) + (1-\lambda)f^c(x_1)$ is immediate if either $x_1 \leqslant \tilde{z}$ or $x_0 \geqslant \tilde{z}$, so assume $x_0 < \tilde{z} < x_1$. Note that by concavity $f^c(x_1) = f(x_1) \leqslant f'_+(\tilde{z})(x_1 - \tilde{z}) + f(\tilde{z}) \leqslant k(x_1 - \tilde{z}) + k\tilde{z} = kx_1$. If $x_\lambda \in (x_0, \tilde{z})$, then:

$$f^c(x_\lambda) = kx_\lambda = k\lambda x_0 + k(1-\lambda)x_1 \geqslant \lambda f^c(x_0) + (1-\lambda)f^c(x_1).$$

If $x_\lambda \in (\tilde{z}, x_1)$, then note that we have

$$\begin{cases} f^c(x_0) = kx_0, \\ f^c(x_1) \leqslant kx_1, \\ f^c(\tilde{z}) = k\tilde{z}, \\ x_0 < \tilde{z} < x_1, \end{cases} \implies \frac{f^c(x_1) - f^c(x_0)}{x_1 - x_0} \geqslant \frac{f^c(x_1) - f^c(\tilde{z})}{x_1 - \tilde{z}}.$$

But this states that the slope of the line through $(\tilde{z}, f^c(\tilde{z}))$ and $(x_1, f^c(x_1))$ is less than the slope of the line through $(x_0, f^c(x_0))$ and $(x_1, f^c(x_1))$. Since $f^c(x_\lambda)$ lies above the former line (by concavity), it must also lie above the latter line. □

Recall that:

$$\Psi(x) = \begin{cases} 0, & x < L_T^g, \\ x - L_T^g, & L_T^g \leqslant x \leqslant \frac{L_T^g}{\alpha}, \\ (1-\delta\alpha)x - (1-\delta)L_T^g, & x > \frac{L_T^g}{\alpha}. \end{cases} \tag{A.2}$$

Note that $\Psi(x)$ is concave and nonnegative on $[L_T^g, \infty)$, and therefore $U(\Psi(x))$ is concave on $[L_T^g, \infty)$ since U is concave and increasing on $[0, \infty)$.

Lemma A.2. *Let $f(x) = U(\Psi(x))$. Then the concave envelope of f is given by (A.1) with:*

$$\tilde{z} = \begin{cases} \frac{L_T^g}{1-\gamma}, & 1-\alpha > \gamma, \\ \frac{(1-\delta)L_T^g}{(1-\delta\alpha)(1-\gamma)}, & (1-\delta\alpha) > 1-\alpha, \\ \frac{L_T^g}{\alpha}, & \gamma \geqslant (1-\alpha) \geqslant (1-\delta\alpha)\gamma. \end{cases} \tag{A.3}$$

$$k = \begin{cases} \gamma(\tilde{z} - L_T^g)^{\gamma-1}, & 1-\alpha > \gamma, \\ \gamma(1-\delta\alpha)((1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g)^{\gamma-1}, & (1-\delta\alpha)\gamma > 1-\alpha, \\ (1-\alpha)^\gamma(\tilde{z})^{\gamma-1}, & \gamma \geqslant 1-\alpha \geqslant (1-\delta\alpha)\gamma. \end{cases} \tag{A.4}$$

Proof. The first two cases are handled similarly. One solves $\tilde{z}f'(\tilde{z}) = f(\tilde{z})$ for $\tilde{z}$ to obtain the given formulas, and verifies that one has $\tilde{z} \in (L_T^g, \frac{L_T^g}{\alpha})$ in the first case, and $\tilde{z} \in (\frac{L_T^g}{\alpha}, \infty)$ in the second case (thus f is differentiable at $\tilde{z}$). Setting $k = \frac{f(\tilde{z})}{\tilde{z}} > 0$ gives the above values, and immediately yields that conditions 1, 2, and 4 of Lemma A.1 are satisfied. $f(x) \leqslant kx$ is automatic on $[0, L_T^g]$, and holds by concavity on $[L_T^g, \tilde{z}]$ since there $f(x) \leqslant f'(\tilde{z})(x - \tilde{z}) + f(\tilde{z}) = kx$. The third case is only slightly more complicated. For the stated values of $\tilde{z}$ and k, one again immediately has conditions 1,2, and 4, of Lemma A.1, and that $k = \frac{f(\tilde{z})}{\tilde{z}}$. The fact that $\gamma \geqslant 1-\alpha$ then also implies that $k \leqslant f'_-(\tilde{z})$, and thus k is a supergradient of the concave function f on $[L_T^g, \infty)$. The remainder of the result follows as in the previous cases. □

The fully protected case is slightly more difficult. However, Lemma A.1 can still be applied after noting that the concave envelope of $f + a$ is $f^c + a$ for any constant a.

Lemma A.3. *Let $f(x) = U(\widehat{\Psi}(x)) + \lambda(L_T^g)^\gamma$. Then the concave envelope of f is given by f^c where f^c is as in (A.1) with:*

i) $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1} = f'(\tilde{z})$, where $\tilde{z}$ is the unique solution to (29) *when $\lambda > \frac{\gamma+\alpha-1}{\alpha} \cdot \left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$.*

ii) $k = \gamma(1-\delta\alpha)((1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g)^{\gamma-1} = f'(\tilde{z})$, where $\tilde{z}$ is the unique solution to (30) *when $\lambda < \frac{\gamma(1-\delta\alpha)+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$.*

iii) $\tilde{z} = \frac{L_T^g}{\alpha}$, and $k = \alpha\left[\left(\frac{1-\alpha}{\alpha}\right)^\gamma + \lambda\right](L_T^g)^{\gamma-1}$ when $\frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1} \leqslant \lambda \leqslant \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$

Proof. i) Elementary calculus shows that there is an unique solution to (29) in $(L_T^g, \frac{L_T^g}{\alpha})$ under the stated conditions on the parameters. For this $\tilde{z}$, $\tilde{z}f'(\tilde{z}) = f(\tilde{z})$ (this is how (29) was defined), and $k = f'(\tilde{z}) = \frac{f(\tilde{z})}{\tilde{z}} = \gamma(\tilde{z} - L_T^g)^{\gamma-1} > 0$. By definition f is concave on $[\tilde{z}, \infty)$. $f(x) \leqslant kx$ on $[L_T^g, \tilde{z}]$ by concavity, and then (since $f(0) = 0$, and $kL_T^g \geqslant f(L_T^g)$) we also have $f(x) \leqslant kx$ on $(0, L_T^g]$ by the convexity of f on this interval.

ii) The proof is similar to i).

iii) With $\tilde{z}$ and k defined as in the statement, one can verify that $k = \frac{f(\tilde{z})}{\tilde{z}}$, and the conditions on the parameters imply that $0 < k \in [f'_+(\tilde{z}), f'_-(\tilde{z})]$, so that k is in the superdifferential of the concave function f restricted to $[L_T^g, \infty)$. Thus $f(x) \leqslant k(x - \tilde{z}) + f(\tilde{z}) = kx$ on $[L_T^g, \tilde{z}]$. Convexity of f on $[0, L_T^g]$ then implies $f(x) \leqslant kx$ and lemma A.1 applies.

□

B. Closed-form Expressions of Conditional Expectations

Proposition B.1. *For the process $\xi_{t,T}$ defined in* (9) *and the price density process ξ_t defined in* (8)*, we have the following formulas:*

$$\mathbb{E}\left[\xi_{t,T}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\Big|\mathcal{F}_t\right] = e^{-r(T-t)}\Phi[d_{1,t}(\beta^*)], \tag{B.1}$$

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\Big|\mathcal{F}_t\right] = e^{-r(T-t)}\frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]}\Phi[d_{2,t}(\beta^*)], \tag{B.2}$$

$$\begin{aligned}\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq c\beta^*\}}\Big|\mathcal{F}_t\right] &= e^{-r(T-t)}\frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]}\Phi[d_{2,t}(c\beta^*)],\\ &= c^{\frac{1}{\gamma-1}}e^{-r(T-t)}\frac{\phi[d_{1,t}(c\beta^*)]}{\phi[d_{2,t}(c\beta^*)]}\Phi[d_{2,t}(c\beta^*)].\end{aligned} \tag{B.3}$$

Proof. We rewrite $\xi_{t,T}$ as follows:

$$\xi_{t,T} = \exp\left[-(r+\frac{\zeta^2}{2})(T-t) + \zeta\sqrt{T-t}\cdot y\right], \text{ where } y = -\frac{W_T - W_t}{\sqrt{T-t}} \sim N(0,1).$$

Then, for equation (B.1), we note that $\xi_t\xi_{t,T} \leq \beta^*$ if and only if

$$y \leq \frac{\ln\beta^* - \ln\xi_t + (r+\frac{1}{2}\zeta^2)(T-t)}{\zeta\sqrt{T-t}} = d_{1,t}(\beta^*) + \zeta\sqrt{T-t}.$$

Therefore,

$$\mathbb{E}\left[\xi_{t,T}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\Big|\mathcal{F}_t\right] = \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}}\exp\left[-(r+\frac{\zeta^2}{2})(T-t)+\zeta\sqrt{T-t}\cdot y\right]\exp\left[-\frac{1}{2}y^2\right]dy$$

$$
\begin{aligned}
&= e^{-r(T-t)} \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(y-\zeta\sqrt{T-t})^2\right] dy \\
&= e^{-r(T-t)} \Phi[d_{1,t}(\beta^*)].
\end{aligned}
$$

For equation (B.2), we note that $\xi_t \xi_{t,T} \le \beta^*$ if and only if $y \le d_{1,t}(\beta^*) + \zeta\sqrt{T-t}$, and thus,

$$
\begin{aligned}
&\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \mathbf{1}_{\{\xi_t\xi_{t,T}\le\beta^*\}}\Big|\mathcal{F}_t\right] \\
=&\left(\frac{\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-(r+\frac{\zeta^2}{2})(T-t)\frac{\gamma}{\gamma-1} + \zeta\frac{\gamma}{\gamma-1}\sqrt{T-t}\cdot y\right] \exp\left[-\frac{1}{2}y^2\right] dy \\
=&e^{-r(T-t)} \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(y-\zeta\frac{\gamma}{\gamma-1}\sqrt{T-t})^2\right] dy \\
&\times\left(\frac{\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \exp\left\{r(T-t)\frac{1}{1-\gamma} + \frac{\zeta^2(T-t)}{2}\left[\frac{\gamma}{1-\gamma} + \left(\frac{\gamma}{1-\gamma}\right)^2\right]\right\} \\
=&e^{-r(T-t)} \frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]} \Phi[d_{2,t}(\beta^*)].
\end{aligned}
$$

Finally, for equation (B.3), we immediately obtain from equation (B.2) that

$$
\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \mathbf{1}_{\{\xi_t\xi_{t,T}\le c\beta^*\}}\Big|\mathcal{F}_t\right] = e^{-r(T-t)} \frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]} \Phi[d_{2,t}(c\beta^*)].
$$

In addition, we have

$$
\begin{aligned}
\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \mathbf{1}_{\{\xi_t\xi_{t,T}\le c\beta^*\}}\Big|\mathcal{F}_t\right] &= c^{\frac{1}{\gamma-1}} \mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{c\beta^*}\right)^{\frac{1}{\gamma-1}} \mathbf{1}_{\{\xi_t\xi_{t,T}\le c\beta^*\}}\right] \\
&= c^{\frac{1}{\gamma-1}} e^{-r(T-t)} \frac{\phi[d_{1,t}(c\beta^*)]}{\phi[d_{2,t}(c\beta^*)]} \Phi[d_{2,t}(c\beta^*)],
\end{aligned}
$$

where the last step is due to equation (B.2) again. □

C. Implementation of the three-step procedure in the proof of Proposition 4.1 for case A1

From (26), the optimal terminal portfolio is given by

$$
\begin{aligned}
X_T^*(\beta^*) = x^*(\beta^*\xi_T) = &\left[\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + L_T^g\right] \mathbf{1}_{\{m/\beta^*<\xi_T\le k/\beta^*\}} + \frac{L_T^g}{\alpha} \mathbf{1}_{\{(1-\delta\alpha)m/\beta^*\le\xi_T\le m/\beta^*\}} \\
&+ \left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1-\delta\alpha}\right] \mathbf{1}_{\{\xi_T<(1-\delta\alpha)m/\beta^*\}},
\end{aligned}
$$

which is the expression in (35).

In addition, the optimal portfolio value at time t, $t \in [0,T)$, $X_t^* = \xi_t^{-1}\mathbb{E}[\xi_T x^*(\beta^*\xi_T)|\mathcal{F}_t] = \mathbb{E}[\xi_{t,T} x^*(\beta^*\xi_t\xi_{t,T})|\mathcal{F}_t]$, and it can be computed as the sum of the following five items:

(1)

$$\begin{aligned}
&\mathbb{E}\left[\xi_{t,T}\left(\frac{\beta^*\xi_t\xi_{t,T}}{\gamma}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{m/\beta^*<\xi_t\xi_{t,T}\le k/\beta^*\}}\right]\\
&=\mathbb{E}\left[\xi_{t,T}\left(\frac{k\xi_t\xi_{t,T}}{\gamma\frac{k}{\beta^*}}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\le k/\beta^*\}}\right]-\mathbb{E}\left[\xi_{t,T}\left(\frac{k\xi_t\xi_{t,T}}{\gamma\frac{k}{\beta^*}}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\le m/\beta^*\}}\right]\\
&=e^{-r(T-t)}\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\left(\Phi[d_{2,t}(k/\beta^*)]-\Phi\left[d_{2,t}\left(m/\beta^*\right)\right]\right),
\end{aligned}$$

(2)

$$\begin{aligned}
&\mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{m/\beta^*<\xi_t\xi_{t,T}\le k/\beta^*\}}\right]\\
&=\mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{\xi_t\xi_{t,T}\le k/\beta^*\}}\right]-\mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{\xi_t\xi_{t,T}\le m/\beta^*\}}\right]\\
&=e^{-r(T-t)}L_T^g\left(\Phi[d_{1,t}(k/\beta^*)]-\Phi\left[d_{1,t}\left(m/\beta^*\right)\right]\right),
\end{aligned}$$

(3)

$$\begin{aligned}
&\mathbb{E}\left[\xi_{t,T}\frac{L_T^g}{\alpha}\mathbf{1}_{\{(1-\delta\alpha)m/\beta^*<\xi_t\xi_{t,T}\le m/\beta^*\}}\right]\\
&=\mathbb{E}\left[\xi_{t,T}\frac{L_T^g}{\alpha}\mathbf{1}_{\{\xi_t\xi_{t,T}\le m/\beta^*\}}\right]-\mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{\xi_t\xi_{t,T}\le(1-\delta\alpha)m/\beta^*\}}\right]\\
&=e^{-r(T-t)}\frac{L_T^g}{\alpha}\left(\Phi[d_{1,t}(m/\beta^*)]-\Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right),
\end{aligned}$$

(4)

$$\begin{aligned}
&\mathbb{E}\left[\xi_{t,T}(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_t\xi_{t,T}}{\gamma}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\xi_t\xi_{t,T}\le(1-\delta\alpha)m/\beta^*\}}\right]\\
&=\mathbb{E}\left[\xi_{t,T}(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{k\xi_t\xi_{t,T}}{\gamma\frac{k}{\beta^*}}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\le(1-\delta\alpha)m/\beta^*\}}\right]\\
&=e^{-r(T-t)}(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\left(\Phi[d_{2,t}((1-\delta\alpha)m/\beta^*)]\right),
\end{aligned}$$

(5)

$$\mathbb{E}\left[\xi_{t,T}\frac{(1-\delta)L_T^g}{1-\delta\alpha}\mathbf{1}_{\{\xi_t\xi_{t,T}\le(1-\delta\alpha)m/\beta^*\}}\right]$$

$$=e^{-r(T-t)}\frac{L_T^g}{\alpha}\left(\Phi[d_{1,t}((1-\delta\alpha)m/\beta^*)]\right).$$

From the above, we obtain the expression (33) for $X_t^{\pi^*}$.

To obtain π_t^*, we rewrite $X_t^{\pi^*}=q(t,\xi_t)$, where q is a C^2 function and simply take the first-order derivative $\frac{\partial q(t,\xi_t)}{\partial \xi_t}$. In this step, we also use the following fact

$$\frac{d}{dx}\left[\frac{\Phi(x)}{\phi(x)}\right]=\frac{\phi^2(x)-\Phi(x)\phi(x)\cdot(-x)}{\phi^2(x)}=1+x\frac{\Phi(x)}{\phi(x)}.$$

After tedious, but straightforward calculation and introducing the function $K(\beta)$ defined in (31), we have (34).

D. Non-negativity of $\pi_t^*(\beta^*)$ and $\widehat{\pi}(\widehat{\nu})$

We know that for both the defaultable and protected policies, the optimal investment strategies $\pi_t^*(\beta^*)$ and $\widehat{\pi}(\widehat{\nu})$ share the same expressions but they differ from each other in terms of the tangent point $\tilde{z}$, the slope of tangent line k, and the entry condition regarding the parameters for the three distinct cases. as shown in Propositions 4.1 and 4.2. Below we only show the non-negativity of $\pi_t^*(\beta^*)$ because that of $\widehat{\pi}(\widehat{\nu})$ follows in the same manner.

Case A1. In this case, $k<(1-\delta\alpha)m<m$ and $\pi_t^*(\beta^*)=\frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(a_1+a_2+a_3+a_4+a_5)$ as given in (34) with explicit expressions for a_1,a_2,a_3,a_4 and a_5 defined there. We begin with a_1, the second term in a_2 and the first term in a_3 to get

$$\begin{aligned}
&\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}K(k/\beta^*)-\left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}}K\left(m/\beta^*\right)-\phi\left[d_{1,t}\left(m/\beta^*\right)\right]+\frac{L_T^g}{\alpha}\phi\left[d_{1,t}\left(m/\beta^*\right)\right]\\
=&\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}K(k/\beta^*)-\left(\frac{1}{\alpha}-1\right)L_T^g\frac{\zeta\sqrt{T-t}}{1-\gamma}\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\Phi[d_{2,t}(m/\beta^*)]\\
\geq&\frac{\zeta\sqrt{T-t}}{1-\gamma}\left\{\left(\tilde{z}-L_T^g\right)\frac{\phi\left[d_{1,t}\left(k/\beta^*\right)\right]}{\phi[d_{2,t}(k/\beta^*)]}\Phi[d_{2,t}(k/\beta^*)]-\left(\frac{1}{\alpha}-1\right)L_T^g\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\Phi[d_{2,t}(m/\beta^*)]\right\}\\
=&\frac{\zeta\sqrt{T-t}}{1-\gamma}L_T^g\left(\frac{1}{\alpha}-1\right)\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\left\{\Phi[d_{2,t}(k/\beta^*)]-\Phi[d_{2,t}(m/\beta^*)]\right\}\geq 0,
\end{aligned}$$

where the first equality follows from the definition of $K(\cdot)$ as given in (31), the first inequality follows by dropping some positive parts, the third equality follows by changing k/β^* to m/β^* in $\frac{\phi[d_{1,t}(m/\beta^*)]}{\phi[d_{2,t}(m/\beta^*)]}$ using the formula in Appendix B, and the second inequality follows from the facts that $\Phi(x)$ is an increasing function of x and that $d_{2,t}(\beta)$ is an increasing function of β.

Then we deal with the second term in a_3, a_4 and a_5 to obtain

$$-\phi\left[d_{1,t}\left(m/\beta^*\right)\right]+(1-\delta\alpha)^{-1}\left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}}K\left[(1-\delta\alpha)m/\beta^*\right]+\frac{L_T^g(1-\delta)}{1-\delta\alpha}\phi d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)$$
$$=\frac{\zeta\sqrt{T-t}}{1-\gamma}L_T^g\left(\frac{1}{\alpha}-1\right)\left(\frac{1}{1-\delta\alpha}\right)\frac{\phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]}{\phi[d_{2,t}((1-\delta\alpha)m/\beta^*)]}\Phi[d_{2,t}((1-\delta\alpha)m/\beta^*)]\geq 0,$$

where we simply plug in the definition of $K(\cdot)$.

The remaining term is the first term in a_2 which is obviously positive. Therefore, $\pi_t^*(\beta^*)$ is non-negative in this case.

Case A2. It is obvious all terms in (37) are non-negative.

Case A3. In this case, $\pi^*(\beta^*)$ is given in (40). We begin with c_1, c_2 and the second term in c_3 to get $\frac{\zeta\sqrt{T-t}}{1-\gamma}L_T^g\left(\frac{1}{\alpha}-1\right)\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\left\{\Phi[d_{2,t}(k/\beta^*)]-\Phi[d_{2,t}(m/\beta^*)]\right\}\geq 0$. The remaining term in c_3 is positive. Thus $\pi_t^*(\beta^*)$ is non-negative in this case.

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