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An improved crack propagation model for plain concrete under fatigue loading

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ABSTRACT

Fatigue crack growth phenomenon in concrete is complex in nature and is characterised by various parameters. Important crack growth characterizing parameters must be considered in the analysis for accurate crack propagation and fatigue life prediction. In the present work, an analytical formulation has been developed to predict the propagation of crack in plain concrete members subjected to repetitive nature of loading. Dimensional analysis approach for fatigue crack growth problems has been adopted in conjunction with the theory of intermediate asymptotic for the development of the proposed model. The model has been derived considering the effect of critical energy dissipation in fatigue called fatigue fracture energy which can capture the observed size effect in concrete fatigue. Other important crack growth characterizing parameters considered in the present formulation are, change in energy release rate, maximum energy release rate, initial crack length, tensile strength, and ratio of maximum aggregate size to characteristic size of the structure. Further, the influence of fracture process zone has been incorporated in the proposed formulation through the loading parameter. The developed closed form expression has been calibrated with the available experimental results. The applicability of the mathematical model has been verified using the existing experimental results following both deterministic and statistical approach. The dependence of various governing parameters on fatigue life has been demonstrated through sensitivity study.

1. Introduction

Fatigue phenomenon is prevalent in various concrete structures namely, airport runways, railway bridges, highway pavements, offshore structures, etc. during the service period. Existence of pre-existing flaws triggers the initiation of cracks when subjected to repetitive load cycles and leads subsequently to a sizeable crack, ultimately causing the final fracture. Concrete is a quasi-brittle material and contains a large size inelastic zone called fracture process zone ahead of the crack tip. The formation of fracture process zone (FPZ) in concrete is highly complicated due to the inherent heterogeneous nature. This inelastic zone controls the propagation of cracks in concrete considerably in addition to the other material and geometrical parameters. Such a material behaviour has posed a challenge for the accurate prediction of fatigue life and crack growth rate in concrete.

One of the earliest fracture mechanics based fatigue crack propagation model defining variation of crack growth rate with respect to change in stress intensity factor has been developed by Paris and Erdogan [1]. Paris law can accurately predict the crack propagation behaviour in metallic structures. The existence of large size FPZ restricts the direct application of Paris law on concrete. In

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Nomenclature

ΔG_I	change in energy release rate	D	depth of beam
ΔK_I	stress intensity factor range	d_{max}	maximum aggregate size
σ_t	tensile strength of the material	E	elastic modulus
a	crack length	G_{Imax}	energy release rate corresponding to maximum load
a_c	critical crack length	N	number of load cycles
b	thickness of beam	R	loading ratio
da/dN	crack growth rate	S	span of beam
		U_C	fatigue fracture energy

light of this, attempts have been made by various researchers to modify this law by incorporating additional crack growth influencing parameters. Bazant and Xu [2] have combined the size effect law with the Paris law by manifesting size dependent equivalent fracture toughness in place of fracture toughness. This law describes the increment of crack length per cycle as a power function of the amplitude of a size-adjusted stress intensity factor, defined by the following equations.

$$\frac{da}{dN} = C\left[\frac{\Delta K_I}{K_{IC}}\right]^n \tag{1}$$

$$K_{IC} = K_{If}\left[\frac{\beta}{1+\beta}\right]^{1/2} \tag{2}$$

$$K_{If} = Bf_u\sqrt{d_0}\,\frac{f(\alpha_0)}{c_n} \tag{3}$$

where f_u denotes the tensile strength, $\beta = \frac{d}{d_0}$,d is specimen depth, d_0 has a meaning of transitional size (dimension) of structure. The applicability of above size adjusted Paris law to high strength concrete has been investigated by Bazant and Schell [3] by performing experiments on high strength concrete. Further, it has been recommended that the size adjusted Paris law developed for plain concrete can also be applied for high strength concrete provided the specimen sizes do not vary significantly. Slowik et al. [4] have proposed a fracture mechanics based crack propagation model for plain concrete member under the action of variable amplitude repetitive loading. The effect of load spike applied in between the constant load cycles has been studied by performing experiments on wedge splitting test specimens. Sain and Chandra Kishen [5] have proposed an advanced fatigue crack propagation model for plain and reinforced concrete beam by modifying the model proposed by Slowik et al.[4]. This model incorporates the effect of loading frequency of applied load, loading history and the size effect parameters.

Researchers have attempted to develop analytical models for the prediction of crack propagation under the action of fatigue loading using the fundamental principles of dimensional analysis and the theory of intermediate asymptotics. One of the most earliest fatigue crack growth model using the above concept has been proposed by Barenblatt and Botvina [6]. On this line, various authors [7–9] have derived the closed form expression of the mathematical model for the prediction of crack growth rate in metallic and quasi-brittle materials. Ciavarella et al. [10] have proposed a unified fatigue crack propagation law using the concepts of dimensional analysis. The correlation between the Paris law constants m and C for concrete like materials are found to be considerably different than metals. Further, Carpinteri and Paggi [11] have used dimensional analysis approach and the concepts of complete and incomplete self-similarity on the Paris curve and Wohler curve on the similar line as done by Barenblatt and Botvina [6]. The assumptions of incomplete self-similarity in dimensionless quantities provide a unified description of fatigue. Paggi [12] has developed a crack growth model to depict a clear picture of classical power law being used to predict fatigue behaviour of quasi-brittle materials like concrete. Theoretically, it has been shown that the parameters used in Paris law are dependent on micro-structural size, crack-size and size-scale. Ray and Chandra Kishen [13] have developed a dimensionally homogeneous mathematical equation to predict fatigue crack propagation in plain concrete. This model incorporates different parameters such as, the tensile strength, fracture energy, loading ratio, initial crack length and structural size. Further, this model has been improved in order to predict fatigue crack propagation of plain concrete under the action of variable amplitude loading [14]. Le and Bazant [15] in their study, attempted to explain the physical mechanisms in the Paris law by considering damage accumulation in the cyclic FPZ at the tip of the crack. Based on this theory, it has been concluded that S-N curve must be size dependent for quasi-brittle structure. Further, a modified form of Paris-Erdogan model for the estimation of crack growth rate under the action of fatigue loading for rocks has been proposed by Le et al. [16]. Through experimental and theoretical investigations, the authors have concluded that observed size effect can be explained by incorporating critical energy dissipation for fatigue crack growth. The proposed expression for critical energy dissipation rate for macro-crack fatigue to account for finite size specimen is given below.

$$U_C = U_{C,\infty}\left(\frac{D}{D+D_{0c}}\right) \tag{4}$$

where $U_{C,\infty}$ is the critical energy dissipation per unit growth of the macro-crack in an infinitely large specimen. D_{0C} is the transitional size for cyclic loading and D denotes the structural size.

According to the authors, a larger size fracture process zone is observed in the case of fatigue loading than the monotonic loading.

The model proposed by Le et al. [16] has been modified by Kirane and Bazant [17] for quasi-brittle materials using dimensional analysis approach. Based on the evidences of experimental and numerical study, on contrary to the studies by Bazant and Xu [2] and Le et al. [16] the authors have observed a smaller size fracture process zone under cyclic loading than the monotonic.

A series of studies on the effect of aggregate size has been carried out on the fracture and fatigue behaviour of concrete structures. According to Nallathambi et al. [18], the size and shape of the aggregates considerably affect the crack propagation rate in concrete structures. According to the authors, an increase in maximum size of the coarse aggregate enhances the resistance to crack growth, thereby increasing the fracture toughness. Issa et al. [19] conducted numerous experiments in order to understand the influence of maximum size of the aggregate on different crack growth characterising parameters namely, surface measured crack length, critical energy release rate and fracture toughness. An increasing trend of fracture toughness with the increase in maximum size of the aggregate has been observed in their research. In a study by Simon and Chandra Kishen [20], the effect of fracture process zone has been considered by modelling the bridging force offered by the coarse aggregates in fatigue life estimation of concrete. Further, they have proposed a crack growth model by considering such bridging action provided by the aggregate. In another study by the same authors [21], a multi-scale based analytical model has been proposed utilizing the concepts of dimensional analysis and self similarity in conjunction with the population growth model. In the model, the stress intensity factor has been calculated through multi-scale approach considering both macro and micro-scale.

2. Methodology

To develop any mathematical equation for a given physical problem, the concepts of dimensional analysis in conjunction with the theory of intermediate asymptotic has been found to be suitable and is utilized in the present study. Firstly, dimensional analysis considers the variables those govern the physical phenomenon under consideration and then converts them into non-dimensional form having physical dimension equal to unity. Use of dimensionless numbers is advantageous as it reduces the number of variables which are needed to define a physical problem and provides physical meaning to the parameters if formed properly leading to a better understanding of the phenomenon. Moreover, the use of dimensionless number can reduce the quantity of experimental data required. Secondly, the application of theory of intermediate asymptotic to this dimensionless numbers can eliminate either too small or large terms. However, depending on the physical problem considered, different self-similar solutions can be derived.

3. Mathematical formulation for crack growth rate

In this section, a fracture mechanics based mathematical equation for prediction of crack growth rate in plain concrete member has been derived. The formulation incorporates various material, geometrical and loading parameters which govern the cracking behaviour under repetitive loading. The closed form expression for the crack growth rate represented as (da/dN) under the action of fatigue loading is derived using the theory of dimensional analysis in conjunction with the theory of intermediate asymptotic. The governing loading parameters which influence the fatigue phenomenon are the change in energy release rate (ΔG_I) and energy release corresponding to the upper fatigue load (G_{Imax}). The material properties considered are tensile strength (σ_t), critical energy dissipation for fatigue (U_C). As described by Le et al. [16] and Kirane and Bazant [17], the critical energy dissipation for fatigue called fatigue fracture energy is considered to be the energy dissipation associated with the growth of macro-crack in each load cycle which is equal to the sum of energy dissipations in all nano-scale cracks inside the cyclic FPZ. The physical quantity, (U_C) is defined as critical energy required to propagate a fatigue crack of unit growth in a finite size specimen. Unlike the consideration of fracture energy for the monotonic loading G_f in the existing analytical formulations for the prediction of fatigue crack growth rate, fatigue fracture energy U_C involves the effective size of cyclic FPZ and can capture size effect in concrete. The geometrical quantities considered are crack length a and the ratio of maximum aggregate size to structural size $(\frac{d_{max}}{D})$ of the member. According to various researchers [18–21], the maximum size of the coarse aggregate d_{max} influences crack resistance and in turn affects the crack propagation behaviour.

The governing variables which affect the fatigue crack propagation are listed with their physical dimensions in Table 1.

The mathematical formulation for crack growth rate can be expressed in terms of the governing variables as below.

Table 1
Governing parameters with their dimensions.

Variables	Description	Dimension
ΔG_I	Change in energy release rate	FL^{-1}
G_{Imax}	Energy release rate corresponding to maximum load	FL^{-1}
D	Structural size	L
d_{max}	Maximum size of the aggregate	L
σ_t	Tensile strength	FL^{-2}
U_C	Fatigue fracture energy	FL^{-1}
a	Crack length	L

$$\frac{da}{dN} = \Phi\left(\Delta G_I, G_{Imax}, \frac{d_{max}}{D}, \sigma_t, U_C, a\right) \tag{5}$$

Considering fatigue fracture energy U_C and tensile strength σ_t to have physically independent dimensions and applying Buckingham Π theorem to Eq. (5), crack growth rate can be expressed in terms of non-dimensional parameters as given below,

$$\frac{da}{dN} = \left(\frac{U_C}{\sigma_t}\right)\Phi_1\left(\frac{\Delta G_I}{U_C}, \frac{G_{Imax}}{U_C}, a\frac{\sigma_t}{U_C}, \frac{d_{max}}{D}\right) \tag{6}$$

$$\frac{da}{dN} = \left(\frac{U_C}{\sigma_t}\right)\Phi_1(\Pi_1, \Pi_2, \Pi_3, \Pi_4) \tag{7}$$

where, the non-dimensional quantities are,

$$\Pi_1 = \frac{\Delta G_I}{U_C}, \quad \Pi_2 = \frac{G_{Imax}}{U_C}, \quad \Pi_3 = a\frac{\sigma_t}{U_C}, \quad \Pi_4 = \frac{d_{max}}{D} \tag{8}$$

The expression in Eq. (6) can be further simplified by using the theory of intermediate asymptotic. Using this, the assumptions of complete and incomplete self-similarity are made for various dimensionless parameters given in Eq. (6). The dimensionless parameter Π_1, is a function of energy release rate range and the critical energy dissipation. The assumption of complete self-similarity to this parameter will make the crack growth rate independent of loading parameter ΔG_I. The crack growth rate is strongly dependent on the loading parameter ΔG_I expressed in terms of energy and cannot be eliminated from the crack propagation model. Considering incomplete self-similarity in Π_1 the ratio can be expressed in power law form as below.

$$\frac{da}{dN} = \left(\frac{U_C}{\sigma_t}\right)\left(\frac{\Delta G_I}{U_C}\right)^{\gamma_1}\Phi_2(\Pi_2, \Pi_3, \Pi_4) \tag{9}$$

The dimensionless number Π_2 is a function of G_{Imax}. The parameter G_{Imax} corresponds to the upper limit of a load cycle and shows a strong dependence on crack growth rate. Incomplete self-similarity assumption in dimensionless parameters Π_2 and Π_3 reduces Eq. (9) to the following form.

$$\frac{da}{dN} = \left(\frac{U_C}{\sigma_t}\right)\left(\frac{\Delta G_I}{U_C}\right)^{\gamma_1}\left(\frac{G_{Imax}}{U_C}\right)^{\gamma_2}\left(a\frac{\sigma_t}{U_C}\right)^{\gamma_3} \Phi_3(\Pi_4) \tag{10}$$

where, coefficients γ_1,γ_2 and γ_3 and the function Φ_3 can be obtained using experimental results. In this study, error minimisation technique has been used to obtain the coefficients introduced during the formulation.

The simplified form of the proposed fatigue crack propagation model derived in Eq. (10) is given below.

$$\frac{da}{dN} = U_C^{1-\gamma_1-\gamma_2-\gamma_3}\Delta G_I^{\gamma_1} G_{Imax}^{\gamma_2}\sigma_t^{\gamma_3-1}a^{\gamma_3}\Phi_3 \tag{11}$$

Now using the relation $G_{Imax} = \Delta G_I/(1-R)^2$, Eq. (11) can further re-written as:

$$\frac{da}{dN} = U_C^{1-\gamma_1-\gamma_2-\gamma_3}\sigma_t^{\gamma_3-1}\Delta G_I^{\gamma_1+\gamma_2}(1-R)^{-2\gamma_2}a^{\gamma_3}\Phi_3 \tag{12}$$

where R is the stress ratio.

4. Discussion on various non-dimensional parameters

The model expressed in Eq. (10) has been derived from Eq. (6) utilizing the theory of intermediate asymptotic. If a dimensionless parameter Π_i is too large or too small giving rise to a finite value of Φ, the quantity Π_i becomes non-essential and can be neglected (complete self-similarity). On contrary, if Φ also tends to zero or infinity with Π_i tending to zero or infinity then the parameter cannot be neglected no matter how small or large it is. However, depending on the behaviour of physical problem, sometimes the quantity can be expressed in power law form.

In the proposed model (Eq. (10)), the assumption of incomplete self similarity has been made in the non-dimensional parameters Π_1,Π_2 and Π_3. The quantity Π_1, is a function of energy release rate range (ΔG_I) and the critical energy dissipation in fatigue (U_C). Therefore, the non-dimensional quantity Π_1 provides indication towards the intensity of crack propagation. A smaller value of ΔG_I than U_C is an indication of stable crack propagation. Further, with the increase in ΔG_I a stage reaches when ΔG_I tends to U_C, indicating a state of unstable crack propagation and fracture. In this work, the quantity Π_1 ranges between 0.00003–0.009 based on the experimental results of Shah [22].

The non-dimensional parameter Π_2 has a strong dependence on crack growth rate as it is a function of maximum value of energy release rate (G_{Imax}) which corresponds to the upper limit of a load cycle. The quantity Π_2 considers and explains the effect of varying load amplitude on crack growth rate and fatigue life. The assumption of incomplete self-similarity of the quantity Π_2 has been made for the values ranging between 0.0001–0.01 based on the experimental results of Shah [22].

Considering the dependence of U_C defined for fatigue loading on characteristic length l_{ch} has similar functional form as that of the fracture energy defined for monotonic loading G_F, the quantity l_{ch} can be expressed as $l_{ch} = EU_C/\sigma_t^2$. Thus, the non-dimensional parameter Π_3 turns out to be a function of ratio of crack length and characteristic length. Moreover, the characteristic length describes the fracture process zone ahead of the crack tip and therefore the quantity Π_3 gives an indication of the state of brittleness. Based on the experimental results of Shah [22], the assumption of incomplete self-similarity for Π_3 has been made for the values ranging between 20 and 100.

The dimensionless parameter Π_4 is a function of maximum size of the aggregate d_{max}. The maximum aggregate size has been found to be dependent on characteristic length [23] which describes the fracture process zone ahead of the crack tip. According to Elices and Rocco [23], the characteristic length increases with the increase in the maximum size of the aggregate. Therefore, the parameter Π_4 describes brittle-ductile failure mode depending on the maximum size of aggregate.

The assumption of incomplete self-similarity on various parameters is verified by plotting and analysing experimental results against the non-dimensional parameters. Firstly, logarithm values of crack growth rate (da/dN) from experiments are plotted against the logarithm values of various non-dimensional parameters namely, Π_1,Π_2,Π_3. In the other case, Log(da/dN) is plotted against and

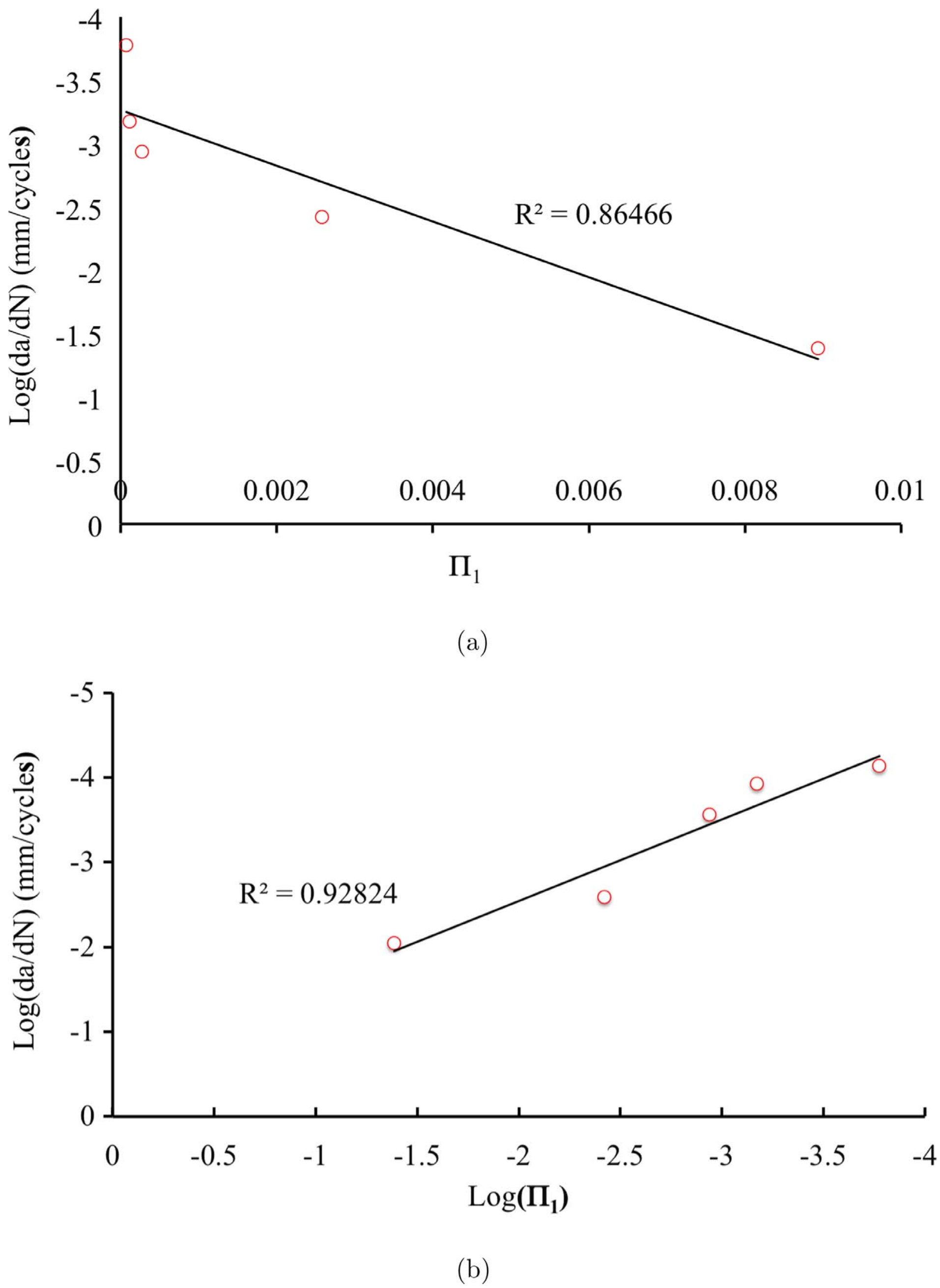
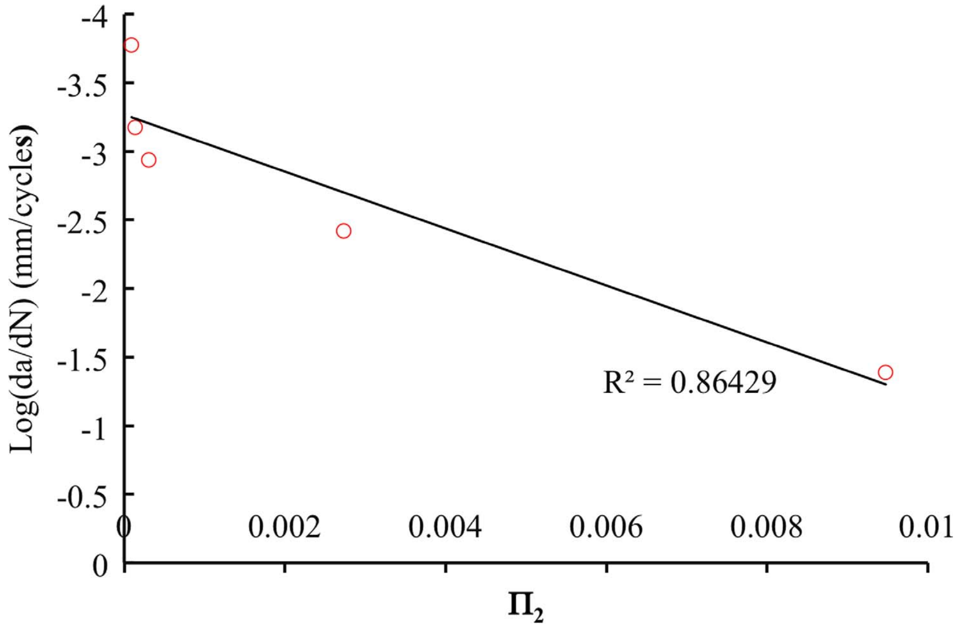


Fig. 1. Power law assessment in Π_1 [22].

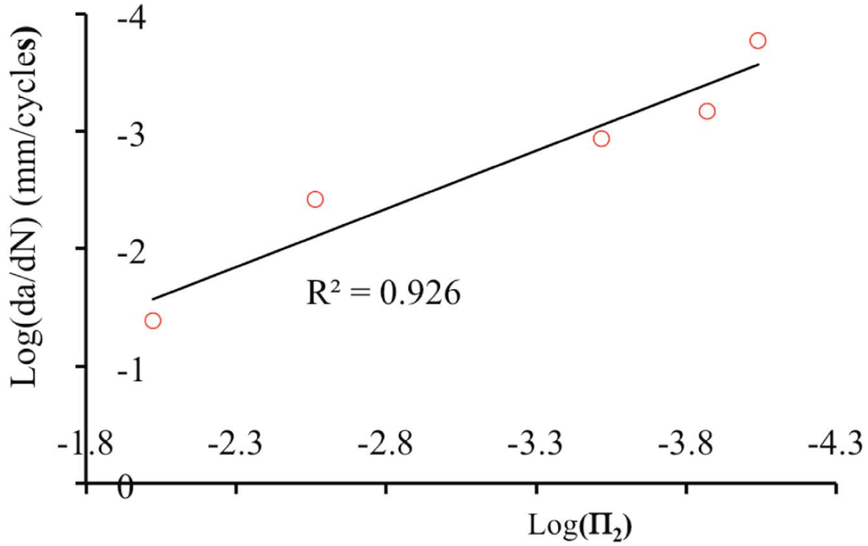
the non-dimensional quantity. Fig. 1 shows the variation of both Log(da/dN) against Log Π_1 and Log(da/dN) against Π_1. Regression coefficient R^2 values based on the linear best fitting of the plots shown in Fig. 1 are 0.92 and 0.86 respectively. A higher regression coefficient in Log(da/dN) versus Log(Π_1) than Log(da/dN) versus Π_1 gives an indication of validity of incomplete self-similarity assumption. Similar plots have been made for Π_2 and Π_3 in Figs. 2 and 3 respectively. In Figs. 2 and 3 also, a higher value of regression coefficient can be observed in Log(da/dN) versus Log(Π_2) and Log(da/dN) versus Log(Π_3).

5. Calibration study of the proposed model

The closed form expression of the proposed model described in Eq. (11) involves several unknown coefficients which have been introduced during the formulation of the mathematical model. In this section, the unknown coefficients introduced in the proposed

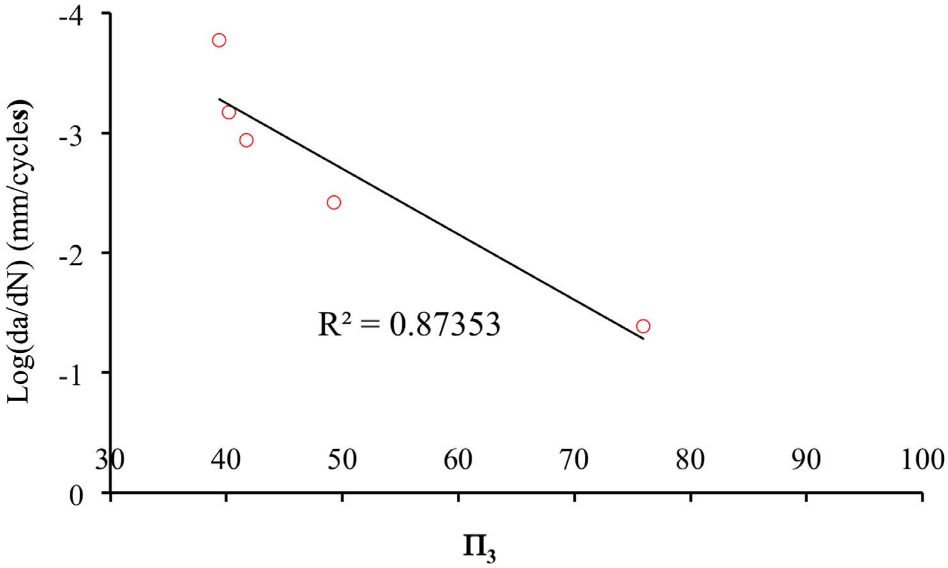


(a)

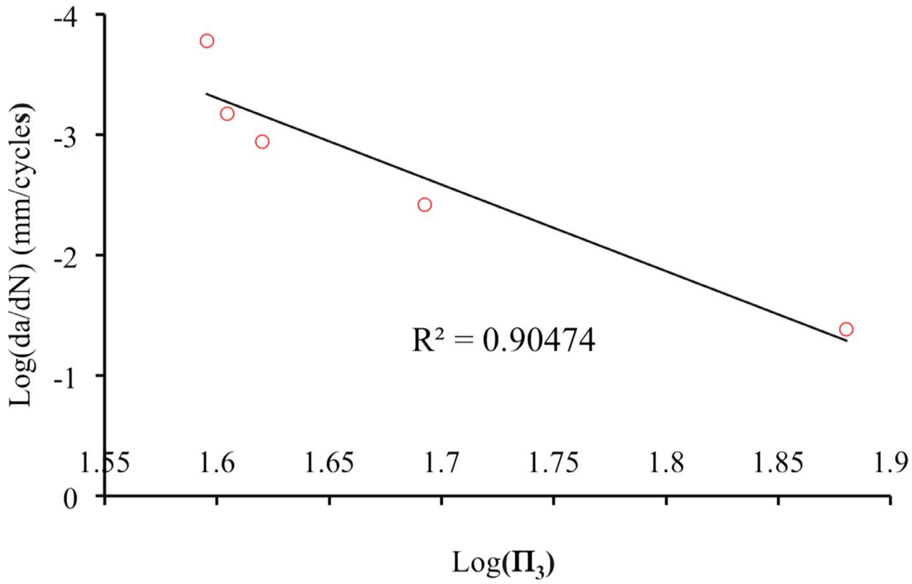


(b)

Fig. 2. Power law assessment in Π_2 [22].



(a)



(b)

Fig. 3. Power law assessment in Π_3 [22].

model are evaluated by using the experimental results of Shah [22] and Bazant and Xu [2]. Shah [22] has conducted three point bending tests on three geometrically similar specimens under varying amplitude sinusoidal loading. The details of specimen geometry are provided in Table 2. The values of modulus of elasticity and tensile strength for the material obtained by Shah are 30,000 N/mm^2 and 3.9 N/mm^2 respectively. Fatigue fracture energy called as critical energy dissipation described for fatigue loading U_C [16] has been calculated by using the experimental plots of Shah [22]. The load versus displacement plots for small, medium and large size beam specimens have been used for the calculation of energy dissipation. Total fatigue fracture energy for each specimen size is obtained by calculating the dissipated energy in each load cycles. The values of U_C for small, medium and large beam specimens are calculated to be 1.54 N/mm, 3.89 N/mm and 12.97 N/mm respectively considering unit crack propagation. For determining the unknown coefficients γ_1,γ_2,γ_3 and Φ_3 the input parameters required are da/dN, G_{Imax}, ΔG_I, and a. The change in energy release rate and maximum energy release rate are calculated using linear elastic approach incorporating the effect of fracture process zone through cohesive force acting in the fictitious crack zone. This is done by following the approach developed by Kumar and Barai [25].

Table 2
Specimen geometry of beams.

Size of the specimen	Dimensions			
	Depth (D) (mm)	Span (S) (mm)	Thickness (b) (mm)	Notch size (a_0) (mm)
Small [22]	76	190	50	15.2
Medium [22]	152	380	50	30.4
Large [22]	304	760	50	60.8
Small [2]	38.1	95.25	38.1	6.35
Medium [2]	76.2	190.5	38.1	12.7
Large [2]	152.4	381	38.1	25.4
Beam [24]	80	320	50	40

According to Kumar and Barai [25], the total stress intensity factor K_I is calculated by using the principle of superposition. The two components of stress intensity factors are due to the applied load (K_I^P) and the cohesive stress acting along the edge of the crack (K_I^C). K_I can be expressed as

$$K_I = K_I^P + K_I^C \tag{13}$$

where, the value of K_I^P can be calculated using linear elastic fracture mechanics. The value of K_I^C is calculated using an analytical expression of cohesive stress intensity factor proposed by Kumar and Barai [25]. The detailed procedure for this calculation has been provided in Appendix A.

An optimization process based on least square error minimisation technique has been adopted to determine the values of the unknown coefficients. The calibrated values of γ_1,γ_2 and γ_3 are 0.064, 1.316 and 1.303 respectively using the experimental results of small beam specimen of Shah [22]. The quantity Φ_3 is a function of $\frac{d_{max}}{D}$. In the experimental study of Shah, the maximum size of aggregate used is 12.5 mm and constant for all the sizes. Based on the calibrated values of Φ_3 and varying $\frac{d_{max}}{D}$, the empirical expression for the quantity Φ_3 is obtained through the best fitting procedure and is given by,

$$\mathrm{Log}\Phi_3 = 198.49\left(\frac{d_{max}}{D}\right)^2 - 49.689\left(\frac{d_{max}}{D}\right) + 2.4842 \tag{14}$$

Bazant and Xu [2] have conducted both monotonic and constant amplitude fatigue testing under flexure on three geometrically

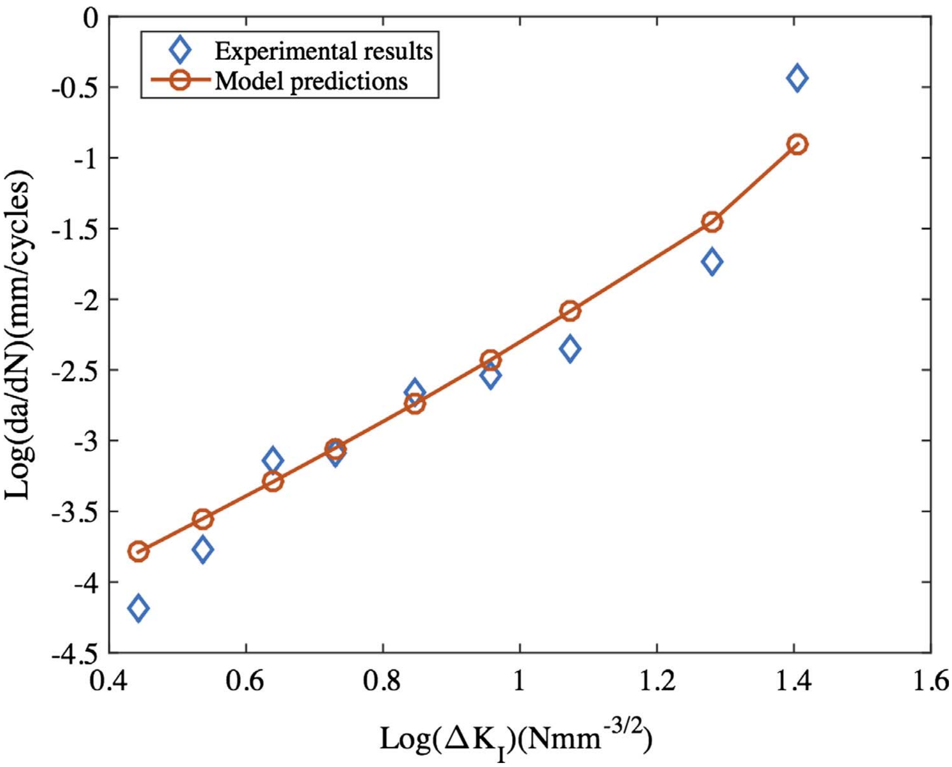


Fig. 4. Variation of crack growth rate (da/dN) as a function of stress intensity factor range (ΔK_I) for medium size specimen [22].

similar specimens. The dimension details of the specimens used in the experimental study are provided in Table 2. The reported values of modulus of elasticity and tensile strength in their study are 27120 N/mm^2 and 2.86 N/mm^2 respectively. The mathematical model derived in Eq. (11) is used along with the calculated coefficients to demonstrate the applicability for other experimental results in the following section.

6. Validation of developed model and discussions

The developed model has been verified with the experimental results of various researchers. Results of Shah [22], Bazant and Xu [2] and Toumi et al. [24] have been used and the predictions of crack growth rate as a function of stress intensity factor range have been made for the developed model. At first, the experimental data points of Shah [22] for medium and large beams have been used for the prediction of the crack growth rate using the proposed model. The logarithm values of crack growth rate Log(da/dN) as a function of stress intensity factor range Log(ΔK_I) are plotted for both proposed model and the experimental results. The above predictions have been made for medium and large size specimens and the results are shown in Figs. 4 and 5 respectively. It may be noted that in the experimental study of Shah [22], the load amplitude is varying in nature and the model predictions shown in the Figs. 4 and 5 are capable of predicting the growth rate with reasonably good accuracy in spite of large scatter in the experimental data. To verify further, the prediction of critical crack length for different load cycles have been made for medium and large size specimens and are plotted in Figs. 6 and 7 along with the experimentally observed values. It can be seen that the model predictions are in good agreement with the experimental results. In summary, the model can predict the crack growth rate with reasonable accuracy for the three specimens with varying structural size and hence, the model is able to capture the effect of fracture process zone and size effect.

Secondly, the experimental results of Bazant and Xu [2] have been used for the comparison with the model predictions. U_C values for experimental data of Bazant and Xu [2] have been taken from the study by Pervaiz and Kishen [26]. The values of U_C for small, medium and large beam specimens are 1.88, 1.96 and 2.83 N/mm respectively considering unit crack propagation. The coefficients γ_1,γ_2 and γ_3 used are 6.35, 0.421 and 1.4 respectively. The crack growth rate has been evaluated as function of stress intensity factor range for small, medium and large sizes using the derived model. The variation in crack growth rate has been plotted in Figs. 8–10 together with the experimentally observed values. The model predictions are found to agree well with the experimental results.

In order to examine further, the experimental study by Toumi et al. [24] has been considered in this section. Toumi et al. [24], in their study have performed experiments on plain concrete beams under three point bending under the action of cyclic loading. The geometry details of the specimen considered in their study has been provided in Table 2. In their study, experiments have been performed with varying stress ratio R. The minimum value of load amplitude P_{min} is kept constant for all the specimens and is taken as

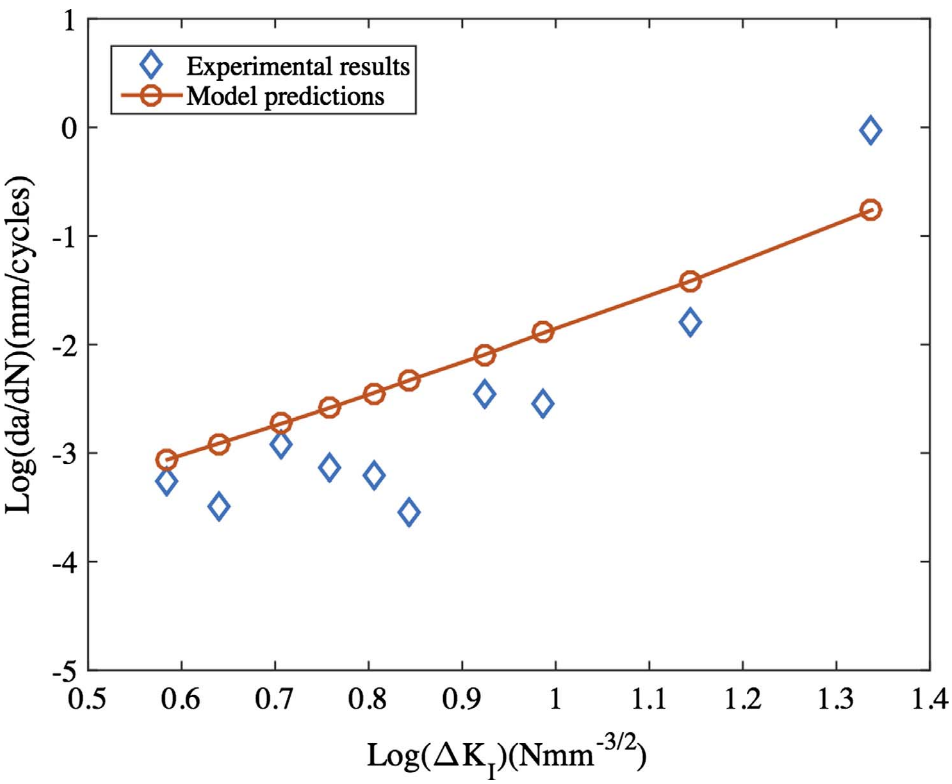


Fig. 5. Variation of crack growth rate (da/dN) as a function of stress intensity factor range (ΔK_I) for large specimen [22].

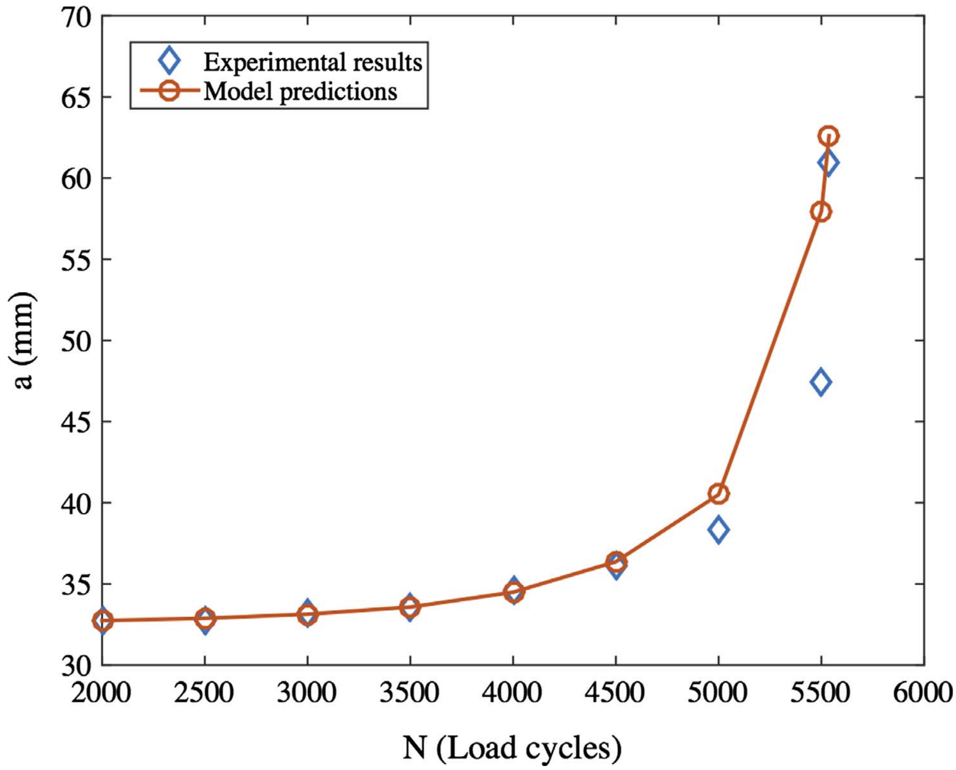


Fig. 6. Variation of crack length (a) as a function of number of load cycles (N) for medium specimen [22].

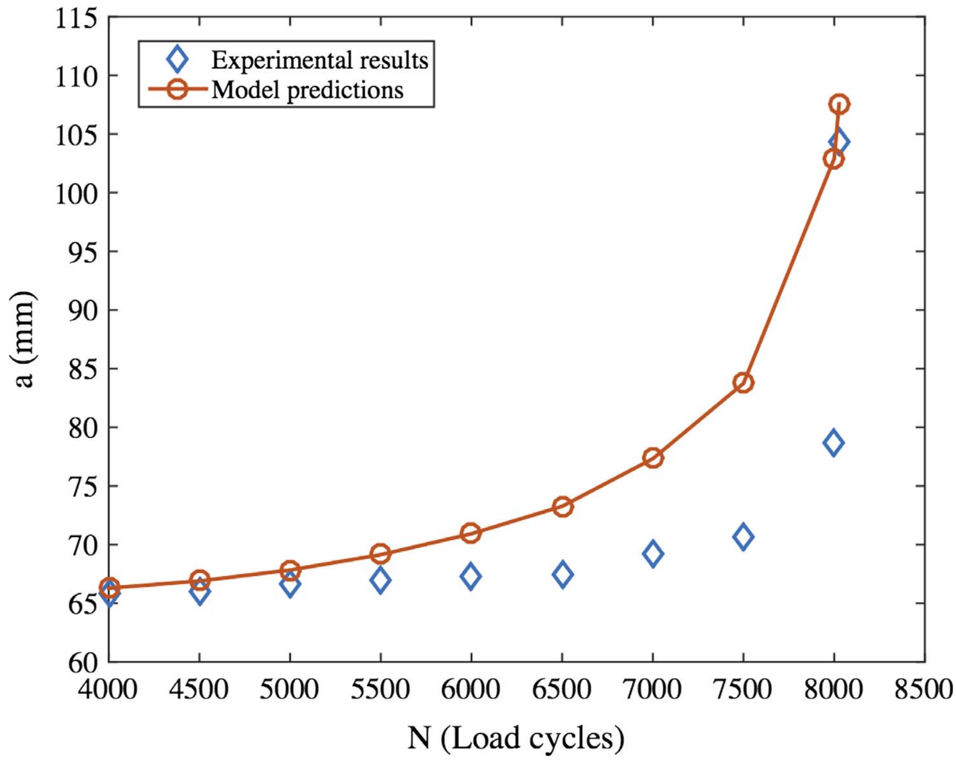


Fig. 7. Variation of crack length (a) as a function of number of load cycles (N) for large specimen [22].

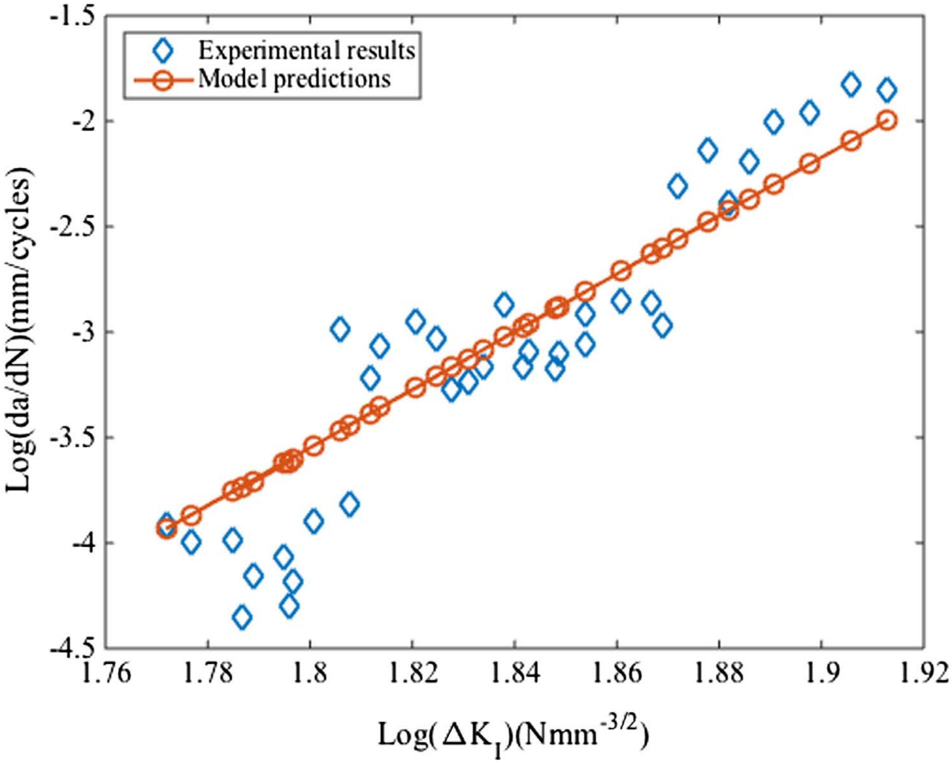


Fig. 8. Variation of crack growth rate (da/dN) as a function of stress intensity factor range (ΔK_I) for small specimen [2].

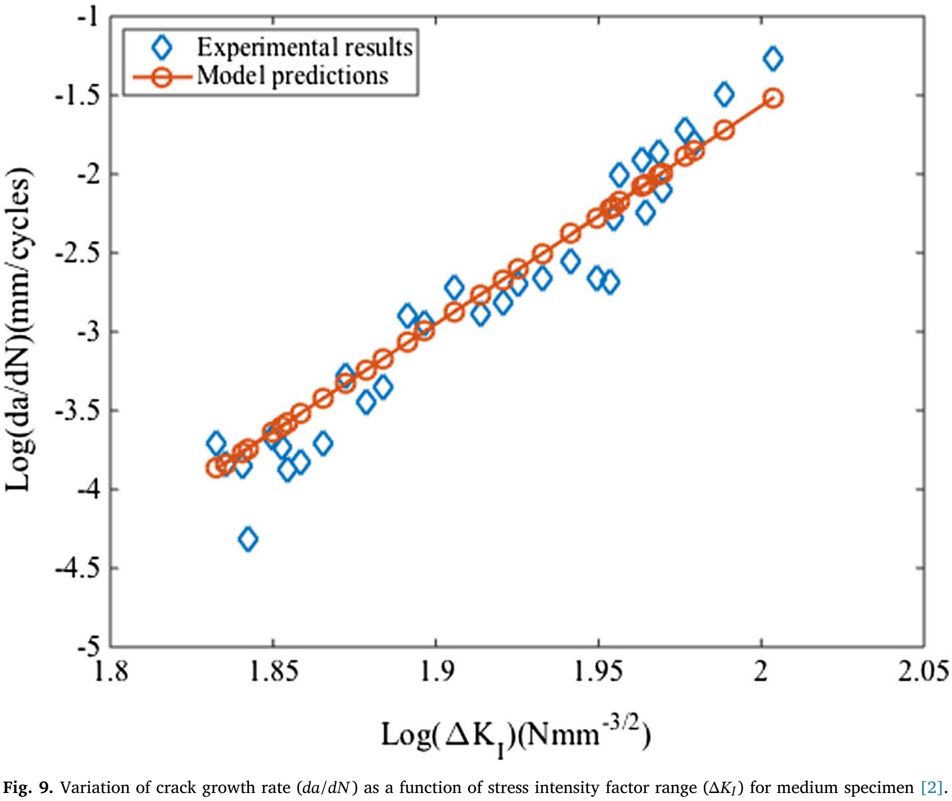


Fig. 9. Variation of crack growth rate (da/dN) as a function of stress intensity factor range (ΔK_I) for medium specimen [2].

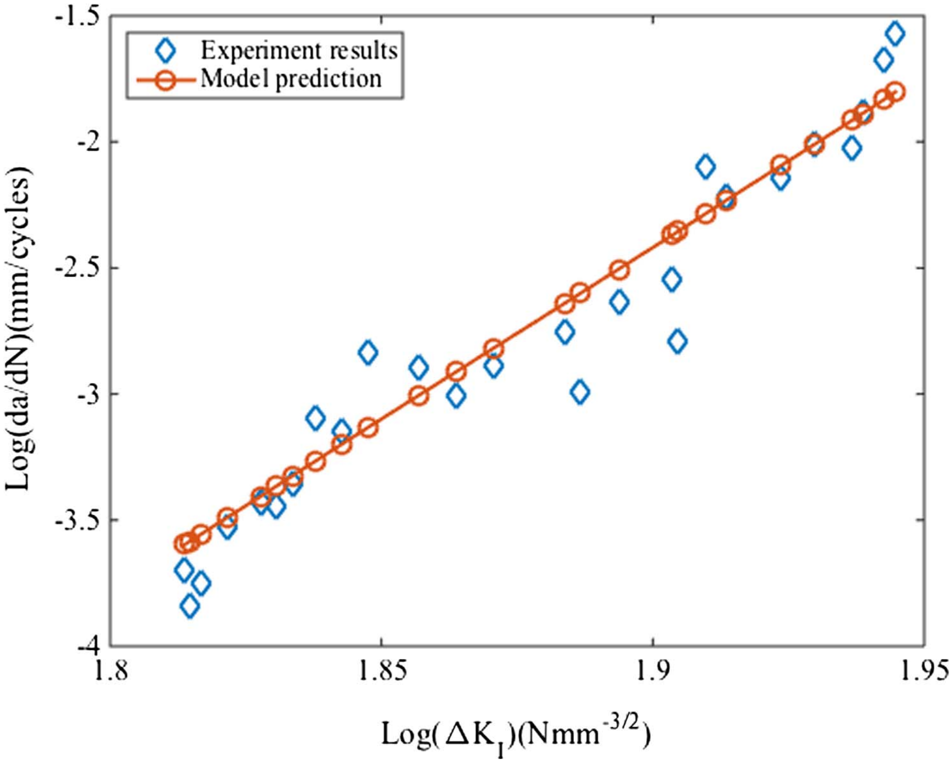


Fig. 10. Variation of crack growth rate (da/dN) as a function of stress intensity factor range (ΔK_I) for large specimen [2].

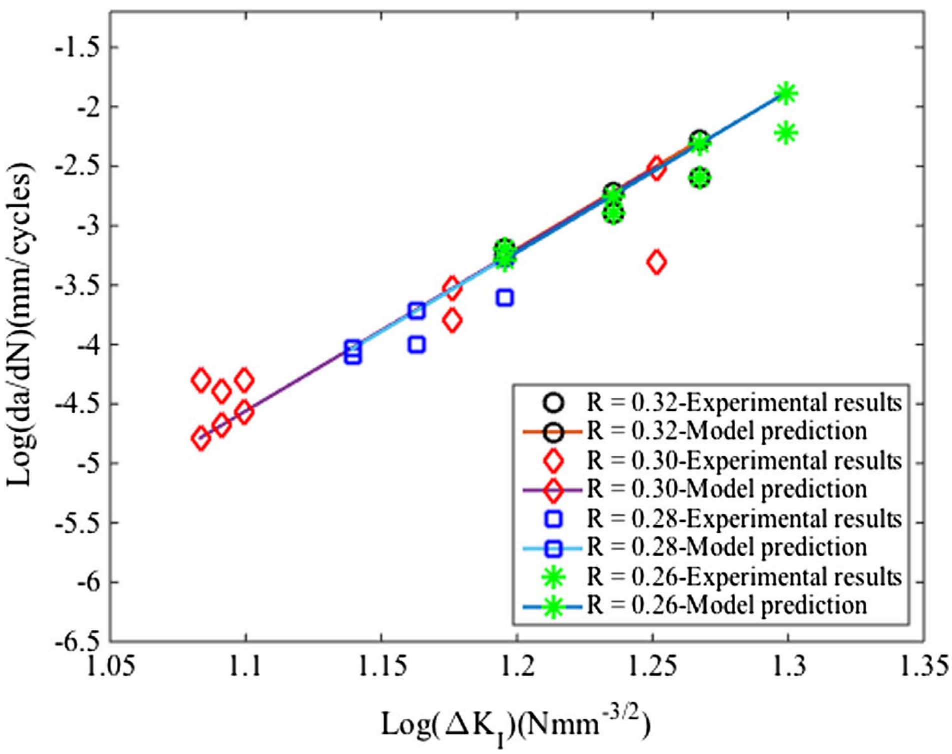


Fig. 11. Variation of crack growth rate (da/dN) as a function of stress intensity factor range (ΔK_I) for varying stress ratio [24].

$0.23P_u$, P_u being the static failure load. Experiments have been carried out with four different stress ratios keeping the maximum load amplitude P_{max} as $0.7P_u$, $0.76P_u$, $0.81P_u$, $0.87P_u$. Crack growth rate has been predicted using proposed analytical model for different stress ratios. The logarithm plots of crack growth rate has been made against stress intensity factor range for both model and experimental results in Fig. 11. The predicted results are found to be in agreement with the experimental results.

7. Statistical analysis

In this section, a statistical approach has been followed to measure the accuracy of the proposed analytical model. Statistical method proposed by Long and Wysockey [27] has been adopted in this study. According to Long and Wysockey [27], the cumulative probability of the ratio (Q_m/Q_e) can be used to measure the prediction exactness of any model. The quantity Q_m denotes the variance of the output parameters predicted from the proposed model and Q_e is variance of the experimental values corresponding to the same parameter. The individual values of the ratio (Q_m/Q_e) for the output quantity are arranged in the ascending order (1,2,3,...,i,...,n). The cumulative probability P of Q_m/Q_e is calculated using the equation provided below.

$$P = \frac{i}{(n+1)} \tag{15}$$

where i = order number given to the ratio Q_m/Q_e and n is the number of data points. Further, the cumulative probability values corresponding to 50% (P_{50}) and 90% (P_{90}) are calculated to measure the accuracy of the proposed model. A value of P_{50} lesser than one implies under prediction of the model and greater than one indicates over prediction. The model predictions are better if the values of P_{50} are closer to unity and the range between P_{50} and P_{90} is smaller. The value of 90% cumulative probability represents the variation for the total observation.

For the above study, the crack growth rate predictions made earlier using the proposed analytical formulation for small, medium and large size specimens of the experimental work of Bazant and Xu [2] and Shah [22] have been considered. The cumulative probability values of Q_m/Q_e for three beam specimens have been calculated using Eq. (15). The variation of cumulative distribution of (Q_m/Q_e) versus the ratio (Q_m/Q_e) have been shown in Fig. 12 for small, medium and large size specimens of experimental work of Bazant and Xu [2]. Similar plot has been made in Fig. 13 for the experimental data of Shah [22]. The cumulative distribution plots for various sizes shown in Figs. 12 and 13 completely describe the distribution of (Q_m/Q_e) wherein, P_{50} and P_{90} values represent the chances of a randomly selected (Q_m/Q_e) value exceeding the corresponding values are 50 and 90% respectively. Therefore, the values of P_{50} and P_{90} closer to unity will have better model predictions.

Further, the cumulative distribution values corresponding to 50% and 90% have been marked in the Figures and the values have

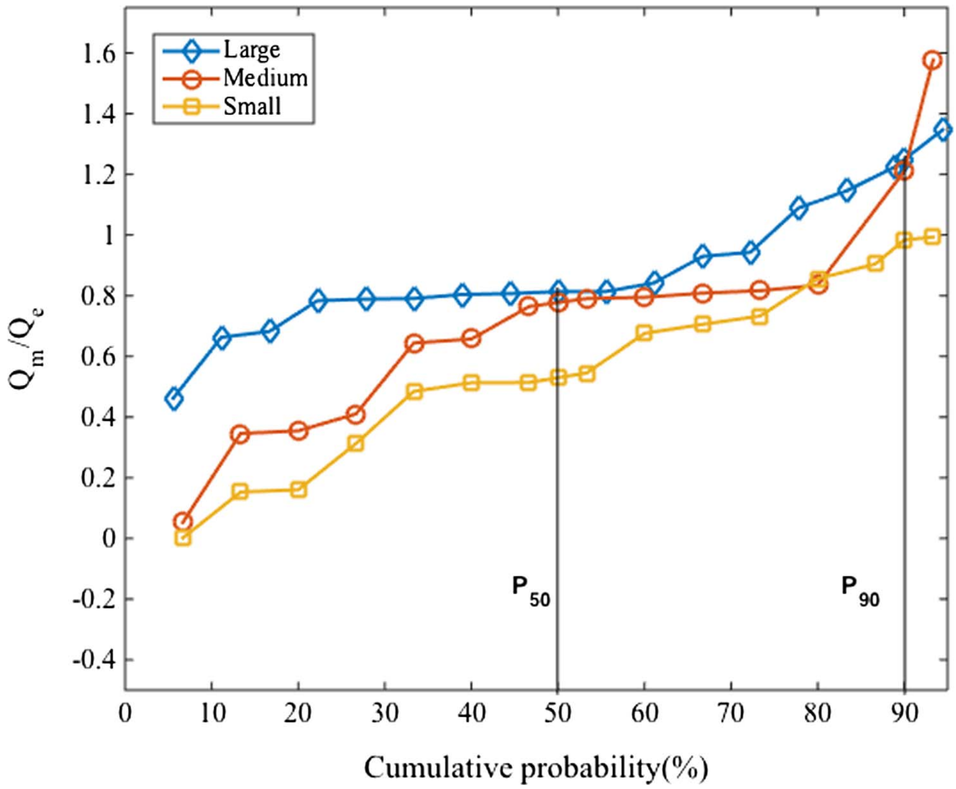


Fig. 12. Cumulative probability plot of (Q_m/Q_e) for small, medium and large sized beam specimens [2].

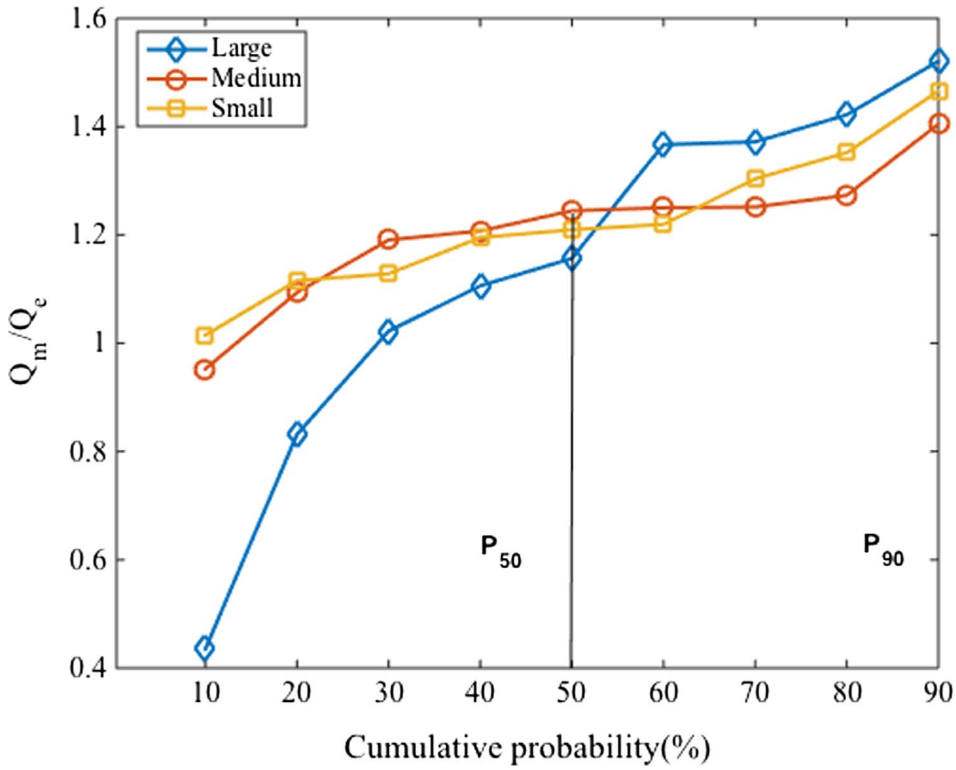


Fig. 13. Cumulative probability plot of (Q_m/Q_e) for small, medium and large sized beam specimens [22].

Table 3
Cumulative probability values for different quantities.

Specimen	Q_m/Q_e at P_{50}	Q_m/Q_e at P_{90}
Beam (Small), Bazant and Xu [2]	0.52	0.99
Beam (Medium), Bazant and Xu [2]	0.78	1.21
Beam (Large), Bazant and Xu [2]	0.81	1.25
Beam (Small), Shah [22]	1.20	1.46
Beam (Medium), Shah [22]	1.24	1.40
Beam (Large), Shah [22]	1.15	1.52

been summarized in Table 3. P_{50} and P_{90} values closer to unity and a smaller range between P_{50} and P_{90} demonstrate better model predictions. It can be observed in Figs. 12 and 13 that the numerical values of P_{50} and P_{90} for various specimens are close to unity confirming the accuracy of the proposed model.

8. Sensitivity analysis

The analytical model proposed for the prediction of crack propagation rate and fatigue life involves various governing parameters which may be deterministic or random in nature. The randomness in different parameters may affect the overall fatigue life. Therefore, in this section, the effect of various parameters involved in the proposed model on fatigue life is studied through sensitivity analysis. A sensitivity study is performed in order to understand the variation in fatigue life N_f corresponding to critical crack length with the randomness in the different input parameters. The sensitivity of an input parameter is measured by computing coefficient of variation using the following equation provided below.

$$p_i = 100*\frac{v_{yi}^2}{v_y} \tag{16}$$

where v_{yi} is defined as the coefficient of variation corresponding to the output quantity considering i^{th} variable as random and remaining as deterministic. v_y is defined as the coefficient of variation corresponding to the output quantity considering all input variables as random. A higher value of coefficient of variation indicates an increased correlation i.e. higher influence of input parameter on the output [5]. The analysis is carried out by computing the fatigue life using the equation given below.

Table 4
Sensitivity coefficient for different variables.

Variables	Mean	Standard deviation	Coefficient of variation
ΔG_I	0.0035	0.00015	11.38
D	152	7.6	82.56
d_{max}	12.5	0.125	3.23
a	30.4	0.17	0.53
a_c	60.93	0.6	1.99
U_C	3.1897	0.031897	0.91
σ_t	3.9	0.39	3.05
G_{Imax}	0.0037	0.00023	20.61

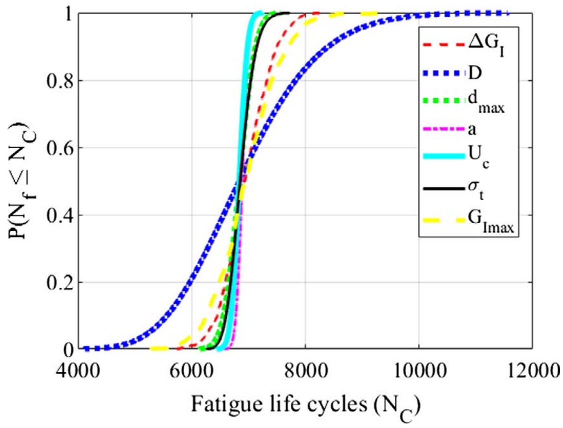


Fig. 14. Probability distribution plots for fatigue life N_f [22].

$$N_f = \int_a^{a_c} \frac{da}{U_C^{1-\gamma_1-\gamma_2-\gamma_3}\sigma_t^{\gamma_3-1}\Delta G_I^{\gamma_1+\gamma_2}(1-R)^{-2\gamma_2}a^{\gamma_3}\Phi_3} \tag{17}$$

where N_f is the fatigue life, a and a_c are the initial and critical crack lengths. The sensitivity coefficients are calculated using medium sized beam of the experimental result of Shah [22]. The input parameters considered for the determination of coefficient of variation are, crack length a, critical crack length a_c, maximum aggregate size d_{max}, structural size D, fatigue fracture energy U_C, change in energy release rate ΔG_I, maximum energy release rate G_{Imax} and tensile strength σ_t. The quantities of v_{yi} and v_y are computed using the randomly generated values of each input variable. Various input quantities along with their statistical parameters and sensitivity coefficients are provided in Table 4. It can be observed that the coefficient of variation for the parameter D is the highest indicating the strongest dependence of structural size on fatigue life. Further, the parameter critical energy dissipation in fatigue, U_C has very less coefficient of variation indicating less sensitiveness towards the fatigue life estimation. Determination of value of U_C from the experimental results is a tedious process. The calculation is dependent on quantities like transition zone for cyclic loading if calculated analytically [16]. A slight variation in the calculated value of U_C will have negligible influence on the fatigue life.

To verify further, the above results are used for the prediction of probability distribution function. In Fig. 14, the ultimate fatigue life N_C has been plotted against the probability of calculated fatigue life N_f which is less than a particular N_C. The parameter for which the curve is steeper, provides an indication that the quantity is less sensitive in comparison to other parameters and can be considered as a deterministic quantity. A similar trend as observed in Table 4 is depicted in Fig. 14 confirming the results.

9. Conclusions

In this study, a mathematical model has been derived for the prediction of crack growth under the action of fatigue loading. This has been calibrated and validated with the available experimental results in the literature to demonstrate its applicability in plain concrete members. The analytical formulation considers important parameters namely, critical energy dissipation in fatigue called fatigue fracture energy (U_C) and the ratio of maximum aggregate size to structural size along with the other conventional crack growth characterizing parameters. According to Le et al. [15] and Kirane and Bazant [17], the parameter critical energy dissipation defined for cyclic loading governs the fatigue crack propagation behaviour and different from critical energy dissipation for the monotonic case G_f. Critical energy in fatigue has been evaluated for geometrically similar beams of different sizes and is incorporated in the proposed model in non-dimensional form. Further, both deterministic and statistical analysis have been followed to verify the performance of the developed model. The results of crack propagation predicted by the model have been compared with the existing

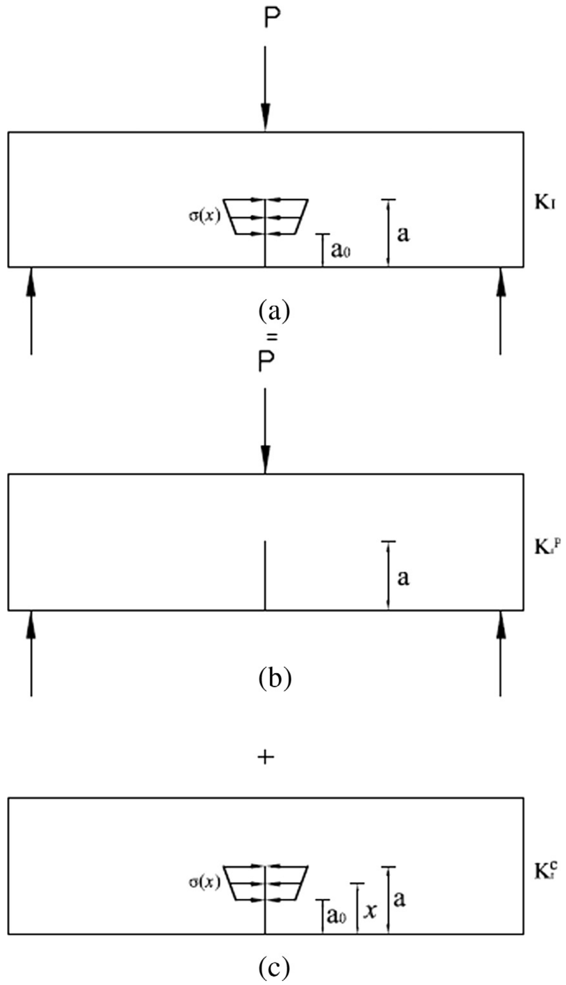


Fig. 15. Superposition of stress intensity factor [25].

Table 5
Coefficients of weight function parameters M_1,M_2 and M_3.

i	a_i	b_i	c_i	d_i	e_i	f_i
1	0.0572011	−0.8741603	4.0465668	−7.89441845	7.8549703	−3.18832479
2	0.4935455	4.43649375	–	–	–	–
3	0.340417	−3.9534104	16.1903942	−16.0958507	14.6302472	−6.1306504

experimental results and are observed to be in good agreement. The influence of each parameter incorporated in the developed model namely, the structural size, fatigue fracture energy, crack length, change in energy release rate, maximum size of the aggregate and tensile strength on fatigue life has been studied through sensitivity analysis. Structural size is found to be the most sensitive parameter to the fatigue life of concrete structures followed by the energy release rate corresponding to maximum load.

Appendix A

In this study, the parameter ΔG_I incorporated in the developed model is calculated using the concept of linear elastic fracture mechanics (LEFM). The relation between stress intensity factor range ΔK_I and ΔG_I is given by,

$$\Delta G_I = \frac{\Delta K_I^2}{E} \tag{18}$$

where ΔK_I is a function of applied load, specimen geometry and crack length. For concrete like quasi brittle materials, the stress intensity factor consists of two components, stress intensity factor due to the applied load (K_I^P) and that due to the cohesive stress acting along the edge of the crack (K_I^C). Total stress intensity factor K_I is given as follow.

$$K_I = K_I^P + K_I^C \tag{19}$$

The value of K_I^C has been computed by adopting the analytical method proposed by Kumar and Barai [25] expressed in the equation provided below.

$$K_I^C = \frac{2}{\sqrt{2\pi a}}\left\{A_1 a\left[2s^{1/2} + M_1 s + \frac{2}{3}M_2 s^{3/2} + \frac{1}{2}s^2\right]\right. \tag{20}$$

Fig. 15 shows a beam subjected to an external load P which has a center crack in it. The two components of stress intensity factors are stress intensity factor due to the applied load (K_I^P) and the cohesive stress acting along the edge of the crack (K_I^C) have been depicted in Fig. 15. According to Kumar and Barai [25], the expression for K_I^C using 3 terms weight function is given by,

$$\left. + A_2 a^2\left[\frac{4}{3}s^{3/2} + \frac{1}{2}M_1 s^2 + \frac{4}{15}M_2 s^{5/2} + \frac{M_3}{6}\{1-(a_0/a)^3 - 3sa_0/a\}\right]\right\} \tag{21}$$

where $A_1 = \sigma(CTOD_c), A_2 = \frac{f_t - \sigma CTOD_c}{a - a_0}$. The values of the weight functions M_1, M_2, M_3 are represented as the function of $\frac{a}{D}$ as provided below. For, $i = 1$ and 3,

$$M_i = \frac{1}{\left(1-\frac{a}{D}\right)^{3/2}}\left[a_i + b_i\left(\frac{a}{D}\right) + c_i\left(\frac{a}{D}\right)^2 + d_i\left(\frac{a}{D}\right)^3 + e_i\left(\frac{a}{D}\right)^4 + f_i\left(\frac{a}{D}\right)^5\right] \tag{22}$$

For $i = 2$.

$$M_i = \left[a_i + b_i\frac{a}{D}\right] \tag{23}$$

Values of a_i, b_i, c_i, d_i, e_i and f_i are given in Table 5. The values of A_1 and A_2 mentioned in Eq. (20) have been calculated using nonlinear softening function proposed by Reinhardt et al. [28] and is provided below.

$$\sigma = f_t\left\{\left[1 + \left(\frac{C_1 w}{w_c}\right)^3\right]exp\left(\frac{-C_2 w}{w_c}\right)^3 - \frac{w}{w_C}(1 + C_1^3)exp(-C_2)\right\} \tag{24}$$

where c_1, c_2, and w_c are the material constants for plain concrete. The values of these constants are $c_1 = 3, c_2 = 6.93$, and $w_c = 160$ mm. The crack opening w is calculated using the following expression proposed by Jenq and Shah [29].

$$w = CMOD_C\left\{\left(1-\frac{a_0}{a_c}\right)^2 + \left(1.081-1.149\frac{a_c}{D}\right)\left[\frac{a_0}{a_c} - \left(\frac{a_0}{a_c}\right)^2\right]\right\}^{1/2} \tag{25}$$

The values of K_I^P have been caluculated using the existing experimental results of Shah [22].

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