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PII: S0378-4371(17)31060-9
DOI: https://doi.org/10.1016/j.physa.2017.10.050
Reference: PHYSA 18764

To appear in: *Physica A*

Received date : 29 March 2017
Revised date : 6 October 2017

Highlights of PHYSA-17575

- The SIPNS model capturing the WOM marketing processes is established.
- The SIPNS model is shown to admit a unique equilibrium.
- The impact of different factors on the equilibrium of the model is illuminated.
- Experiments suggest that the equilibrium is much likely to be globally attracting.
- The influence of different factors on the expected overall profit is ascertained.

The modeling and analysis of the word-of-mouth marketing

Pengdeng Li[a], Xiaofan Yang[a,*], Lu-Xing Yang[a,b], Qingyu Xiong[a], Yingbo Wu[a], Yuan Yan Tang[c]

[a] School of Software Engineering, Chongqing University, Chongqing, 400044, China
[b] Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Delft, GA 2600, The Netherlands
[c] Department of Computer and Information Science, The University of Macau, Macau

Abstract

As compared to the traditional advertising, word-of-mouth (WOM) communications have striking advantages such as significantly lower cost and much faster propagation, and this is especially the case with the popularity of online social networks. This paper focuses on the modeling and analysis of the WOM marketing. A dynamic model, known as the SIPNS model, capturing the WOM marketing processes with both positive and negative comments is established. On this basis, a measure of the overall profit of a WOM marketing campaign is proposed. The SIPNS model is shown to admit a unique equilibrium, and the equilibrium is determined. The impact of different factors on the equilibrium of the SIPNS model is illuminated through theoretical analysis. Extensive experimental results suggest that the equilibrium is much likely to be globally attracting. Finally, the influence of different factors on the expected overall profit of a WOM marketing campaign is ascertained both theoretically and experimentally. Thereby, some promotion strategies are recommended. To our knowledge, this is the first time the WOM marketing is treated in this way.

Keywords: word-of-mouth marketing, overall profit, differential dynamical system, equilibrium, global attractivity

1. Introduction

Promotion is a common form of product sales. The third-party advertising on mass media such as TV and newspaper has long been taken as the major means of promotion. However, this promotion strategy suffers from expensive cost [1, 2]. Furthermore, it has been found that, beyond the early stage of product promotion, the efficacy of advertising diminishes [3]. Word-of-mouth (WOM) communications are a pervasive and intriguing phenomenon. It has been found that satisfied and dissatisfied consumers tend to spread positive and negative comments, respectively, regarding the items they have purchased and used [4, 5]. As compared to positive comments, negative comments are more emotional and, hence, are more likely to influence the receiver's opinion. By contrast, positive comments are more cognitive and more considered [6–9]. The significant role of WOM in product sales is supported by broad agreement among practitioners and academics. Indeed, both positive and negative WOM will affect the purchase decision of potential consumers. Due to striking advantages such as significantly lower cost and much faster propagation, the WOM marketing outperforms the traditional advertising marketing [10, 11]. With the increasing popularity of online social networks such as Facebook, Myspace, and Twitter, the WOM marketing has come to be one of the main forms of product marketing [12].

Currently, the major concern on WOM marketing focuses on finding a set of seeds such that the expected number of individuals activated from this seed set is maximized [13]. Toward this direction, large number of seeding algorithms have been reported [14–23]. Additionally, a number of dynamic models capturing the WOM spreading processes have been suggested [24–34]. However, all the previous work builds on the premise that a single product or a few competing products are on sale. Typically, customers involved in a marketing campaign may purchase multiple products. The ultimate goal of such marketing campaigns is to maximize the overall profit. To achieve the goal, it is

*Corresponding author
Email addresses: 1414797521@qq.com (Pengdeng Li), xfyang1964@gmail.com (Xiaofan Yang), ylx910920@gmail.com (Lu-Xing Yang), Xiong03@equ.edu.cn (Qingyu Xiong), wyb@cqu.edu.cn (Yingbo Wu), yytang@umac.mo (Yuan Yan Tang)

crucial to determine those factors that have significant influence on the overall profit. To our knowledge, so far there is no literature in this aspect.

This paper addresses the modeling and analysis of the WOM marketing for a consistent set of items. First, a dynamic model, which is known as the SIPNS model, that characterizes the WOM marketing processes with both positive and negative comments is established. Second, a measure of the overall profit of a WOM marketing campaign is introduced. Third, the SIPNS model is shown to admit a unique equilibrium, and the equilibrium is figured out. Next, the impact of different factors on the equilibrium of the SIPNS model is expounded through theoretical analysis, and extensive experiments show that the equilibrium is much likely to be globally attracting. Finally, the impact of different factors on the expected overall profit of a WOM marketing campaign is ascertained through both theoretical analysis and simulation experiment. On this basis, some promotion strategies are recommended. To our knowledge, this is the first time the WOM marketing is modeled and analyzed in this way.

The subsequent materials are organized as follows. Section 2 describes the SIPNS model, and presents a measure of the overall profit. Section 3 studies the SIPNS model. Section 4 reveals the influence of different factors on the expected overall profit. Finally, Section 5 closes this work.

2. The modeling of the WOM marketing

Suppose a marketer is asked to plan a WOM marketing campaign for promoting a batch of items, with the goal of achieving the maximum possible overall profit. To achieve the goal, the marketer needs to establish a mathematical model for the WOM marketing campaign and, thereby, to make a comparison among different marketing strategies in terms of the overall profit. This section is devoted to the modeling of the WOM marketing.

2.1. The target market and its state

Later on, it will be seen that the expected overall profit of a WOM marketing campaign is closely related to the WOM marketing process. Now, let us establish a dynamic model characterizing the WOM marketing processes.

Consider a WOM marketing campaign that starts at time $t = 0$ and terminates at time $t = T$. Define the *target market* for the campaign at time t as the set of all the consumers and potential consumers involved in the campaign at time t. Let $X_M(t)$ denote the size of the target market at time t. Due to the impact of many known or unknown factors, $X_M(t)$ is usually uncertain. Let $M(t)$ denote the expectation of $X_M(t)$. Then,

$$M(t) = \sum_{n=0}^{\infty} n \cdot \Pr\{X_M(t) = n\}, \quad t \in [0, T]. \tag{1}$$

Henceforth, we assume that $M(0) = M_0$.

In what follows, it is assumed that, at any time, every individual in the target market is in one of the four possible states: (a) *susceptible*, which means that the individual hasn't recently purchased any item but tends to purchase one, (b) *infected*, which means that the individual has recently purchased an item but hasn't yet made any comment on it, (c) *positive*, which means that the individual has recently purchased an item and has made a positive comment on it, and (d) *negative*, which means that the individual has recently purchased an item and has made a negative comment on it.

Let $X_S(t)$, $X_I(t)$, $X_P(t)$ and $X_N(t)$ denote the number of susceptible, infected, positive and negative individuals at time t, respectively. Then, the vector

$$\mathbf{X}(t) = (X_S(t), X_I(t), X_P(t), X_N(t)) \tag{2}$$

represents the state of the target market at time t. By the relevant definitions, we have

$$X_S(t) + X_I(t) + X_P(t) + X_N(t) = X_M(t), \quad t \in [0, T]. \tag{3}$$

Due to the impact of a variety of factors, these quantities are all uncertain. Let $S(t)$, $I(t)$, $P(t)$ and $N(t)$ denote the expectation of $X_S(t)$, $X_I(t)$, $X_P(t)$ and $X_N(t)$, respectively.

$$S(t) = \sum_{n=0}^{\infty} n \cdot \Pr\{X_S(t) = n\}, \quad t \in [0, T], \tag{4}$$

$$I(t) = \sum_{n=0}^{\infty} n \cdot \Pr\{X_I(t) = n\}, \quad t \in [0, T], \tag{5}$$

$$P(t) = \sum_{n=0}^{\infty} n \cdot \Pr\{X_P(t) = n\}, \quad t \in [0, T], \tag{6}$$

$$N(t) = \sum_{n=0}^{\infty} n \cdot \Pr\{X_N(t) = n\}, \quad t \in [0, T]. \tag{7}$$

Then, the vector

$$\mathbf{S}(t) = (S(t), I(t), P(t), N(t)) \tag{8}$$

represents the expected state of the target market at time t.

2.2. A dynamic model capturing the WOM marketing processes

For the purpose of establishing a mathematical model for the WOM marketing processes, let us impose a set of statistical hypotheses as follows.

(H_1) Due to the influence of advertising, at any time new individuals enter the target market and become susceptible at the average rate $\mu > 0$. We refer to μ as the *entrance rate*.

(H_2) Due to the loss of interest in shopping, at any time an infected (respectively, a positive, a negative) individual exits from the target market at the average rate $\delta_I > 0$ (respectively, $\delta_P > 0$, $\delta_N > 0$). We refer to δ_I, δ_P and δ_N as the *I-exit rate*, *P-exit rate*, and *N-exit rate*, respectively. Certainly, we have $\delta_P \leq \delta_I \leq \delta_N$.

(H_3) Encouraged by the positive comments, at time t a susceptible individual purchases an item and, hence, becomes infected at the average rate $\beta_P P(t)$, where $\beta_P > 0$ is a constant. We refer to β_P as the *P-infection force*.

(H_4) Discouraged by the negative comments, at time t a susceptible individual exits from the market at the average rate $\beta_N N(t)$, where $\beta_N > 0$ is a constant. We refer to β_N as the *N-infection force*.

(H_5) Due to the desire to express the feeling for the recently purchased item, at any time an infected individual makes a positive (respectively, negative) comment on the item and hence becomes positive (respectively, negative) at the average rate $\alpha_P > 0$ (respectively, $\alpha_N > 0$). We refer to α_P and α_N as the *P-comment rate* and *N-comment rate*, respectively.

(H_6) Due to the shopping desire, an infected (respectively, a positive) individual tends to purchase one more item and hence becomes susceptible at the average rate $\gamma_I > 0$ (respectively, $\gamma_P > 0$). We refer to γ_I and γ_P as the *I-viscosity rate* and *P-viscosity rate*, respectively. Certainly, we have $\gamma_I \leq \gamma_P$.

Fig. 1 demonstrates these hypotheses schematically.

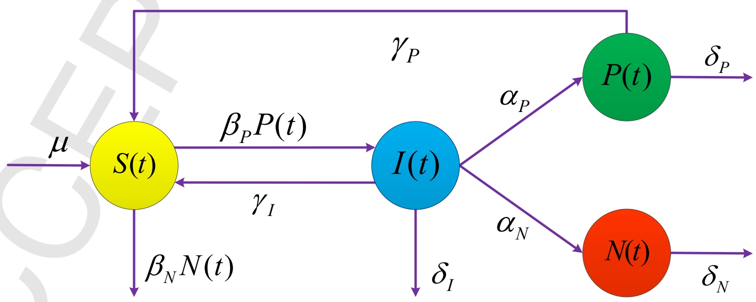

Figure 1. A schematic representation of the hypotheses (H_1)-(H_6).

This collection of hypotheses implies the following differential dynamical system.

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta_P P(t)S(t) - \beta_N N(t)S(t) + \gamma_P P(t) + \gamma_I I(t), & t \in [0, T], \\ \dfrac{dI(t)}{dt} = \beta_P P(t)S(t) - \alpha_P I(t) - \alpha_N I(t) - \gamma_I I(t) - \delta_I I(t), & t \in [0, T], \\ \dfrac{dP(t)}{dt} = \alpha_P I(t) - \gamma_P P(t) - \delta_P P(t), & t \in [0, T], \\ \dfrac{dN(t)}{dt} = \alpha_N I(t) - \delta_N N(t), & t \in [0, T], \end{cases} \tag{9}$$

subject to $M(0) = M_0$. We refer to the system as the *Susceptible-Infected-Positive-Negative-Susceptible model* (the *SIPNS model*, for short). The model captures the expected WOM marketing processes.

2.3. A measure of the overall profit

Based on the SIPNS model, we are ready to measure the overall profit of a WOM marketing campaign. Hereafter we will take the *uniform-profit assumption*: selling an item will bring about an one-unit profit.

It can be seen from the second equation in the SIPNS model (9) that the excepted increment of the number of the individuals who purchase an item in the infinitesimal time horizon $[t, t + dt)$ is $\beta_P P(t)S(t)dt$. Hence, the expected profit gained in this time horizon is $\beta_P P(t)S(t)dt$. It follows that the expected overall profit of the marketing campaign is

$$J = \beta_P \int_0^T P(t)S(t)dt. \tag{10}$$

Naturally, we will take this quantity as a measure of the overall profit of the marketing campaign.

Obviously, the expected overall profit relies the ten model parameters: the entrance rate, the three exit rates, the two comment rates, the two infection rates, and the two viscosity rates. So, the expected overall profit can be written as

$$J = J(\mu, \delta_I, \delta_P, \delta_N, \alpha_P, \alpha_N, \beta_P, \beta_N, \gamma_I, \gamma_P). \tag{11}$$

3. The dynamics of the SIPNS model

The key to the enhancement of the expected overall profit of a WOM marketing campaign is to gain insight into the dynamics of the SIPNS model. This section is dedicated to the study of the dynamics of the SIPNS model.

3.1. The equilibrium

An *equilibrium* of a differential dynamical system is a state of the system such that, starting from the state, the system will always stay in the state. Clearly, the equilibria of a differential dynamical system are the best-understood states of the system. Therefore, the first step toward understanding the dynamics of a differential dynamical system is to determine all of its equilibria. The following result determines all the equilibria of the SIPNS model (9).

Theorem 1. *The SIPNS model (9) admits a unique equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$*, where*

$$\begin{cases} S^* = \dfrac{(\alpha_P + \alpha_N + \gamma_I + \delta_I)(\gamma_P + \delta_P)}{\alpha_P \beta_P}, \\ I^* = \dfrac{\mu \alpha_P \beta_P \delta_N (\gamma_P + \delta_P)}{\alpha_P \beta_P \delta_N (\alpha_N + \delta_I)(\gamma_P + \delta_P) + \alpha_P^2 \beta_P \delta_P \delta_N + \alpha_N \beta_N (\gamma_P + \delta_P)^2 (\alpha_P + \alpha_N + \gamma_I + \delta_I)}, \\ P^* = \dfrac{\mu \alpha_P^2 \beta_P \delta_N}{\alpha_P \beta_P \delta_N (\alpha_N + \delta_I)(\gamma_P + \delta_P) + \alpha_P^2 \beta_P \delta_P \delta_N + \alpha_N \beta_N (\gamma_P + \delta_P)^2 (\alpha_P + \alpha_N + \gamma_I + \delta_I)}, \\ N^* = \dfrac{\mu \alpha_P \alpha_N \beta_P (\gamma_P + \delta_P)}{\alpha_P \beta_P \delta_N (\alpha_N + \delta_I)(\gamma_P + \delta_P) + \alpha_P^2 \beta_P \delta_P \delta_N + \alpha_N \beta_N (\gamma_P + \delta_P)^2 (\alpha_P + \alpha_N + \gamma_I + \delta_I)}. \end{cases} \tag{12}$$

Proof. Let $\mathbf{E} = (S, I, P, N)$ be an equilibrium of the SIPNS model (9). Then,

$$\begin{cases} \mu - \beta_P PS - \beta_N NS + \gamma_P P + \gamma_I I = 0, \\ \beta_P PS - \alpha_P I - \alpha_N I - \gamma_I I - \delta_I I = 0, \\ \alpha_P I - \gamma_P P - \delta_P P = 0, \\ \alpha_N I - \delta_N N = 0. \end{cases} \tag{13}$$

By the third and fourth equations of the system, we get that

$$P = \frac{\alpha_P}{\gamma_P + \delta_P} I, \quad N = \frac{\alpha_N}{\delta_N} I. \tag{14}$$

Substituting the system into the second equation of the system (13) and noticing that $I \neq 0$, we get that $S = S^*$. Substituting this equation and the system (14) into the first equation of the system (13) and simplifying, we derive that $I = I^*$. Substituting this equation into the system (14), we deduce that $P = P^*$ and $N = N^*$. The proof is complete.

It follows from this theorem that, starting from the equilibrium $\mathbf{E}^*$, the SIPNS model will always stay in this equilibrium. Moreover, the location of the equilibrium is determined.

In reality, however, the probability of the event that a differential dynamical system starts from an equilibrium is often vanishingly small. To have a full qualitative understanding of the dynamics of the system, one must be aware of the evolutionary trend of the system when starting from an initial state other than any of the equilibria, and this often involves the stability properties of the equilibrium. An equilibrium is *stable* if, starting from near the equilibrium, the system will always stay near the equilibrium. An equilibrium is *globally attracting* if, starting from any initial state, the system approaches the equilibrium. An equilibrium is *globally stable* if it is stable and globally attracting.

Because of the inherent complexity of the SIPNS model, we failed to prove the stability of its equilibrium, let alone the global stability of the equilibrium. Nevertheless, due to its practical significance and potential application, the SIPNS model is worth further study. Next, let us turn our attention to the study of the SIPNS model through computer simulations, with emphasis on the impact of different factors on the dynamics of the model.

All the subsequent theorems are proved either through direct observation or by applying the following lemma.

Lemma 1. *Let $a, b, c, d > 0$. The following claims hold.*

(a) The function $f_1(x) = ax + \frac{b}{x}$ $(x > 0)$ is strictly decreasing with $x < \sqrt{\frac{b}{a}}$, is strictly increasing with $x > \sqrt{\frac{b}{a}}$, and attains the minimum at $x = \sqrt{\frac{b}{a}}$.

(b) The function $f_2(x) = \frac{ax+b}{cx+d}$ $(x > 0)$ is strictly increasing or strictly decreasing or constant according as $ad > bc$ or $ad < bc$ or $ad = bc$.

Proof. The first claim follows from $f_1'(x) = a - \frac{b}{x^2}$. The second claim follows from $f_2'(x) = \frac{ad-bc}{(cx+d)^2}$.

3.2. The impact of the entrance rate

By Theorem 1, the entrance rate affects the equilibrium of the SIPNS model in the following way.

Theorem 2. *Consider the equilibrium $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ of the SIPNS model (9). The following claims hold.*

(a) S^* is irrelevant to μ.

(b) I^*, P^* and N^* are strictly incresing with μ.

Extensive experiments show that, typically, the entrance rate affects the dynamics of the SIPNS model in the way shown in Fig. 2. In general, it is drawn that, for any entrance rate, the SIPNS model approaches the equilibrium.

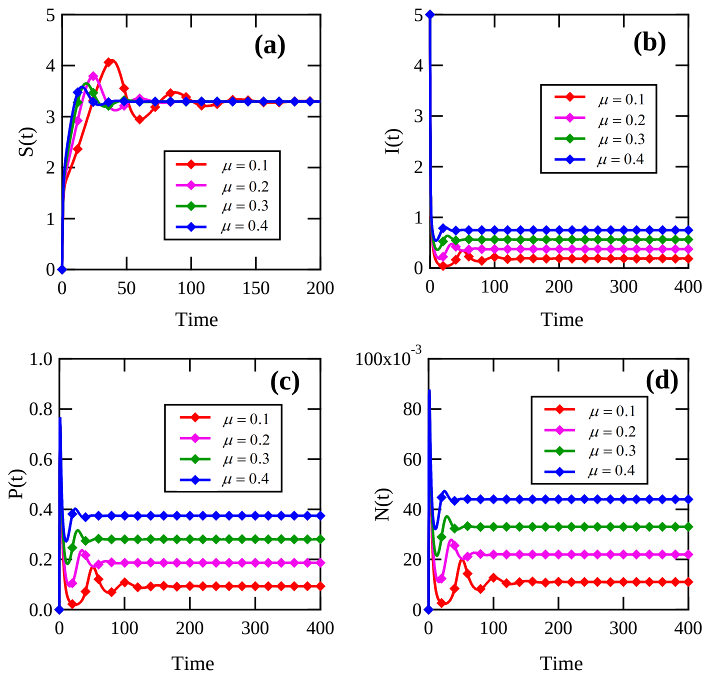

Figure 2. The time plots of $S(t)$, $I(t)$, $P(t)$ and $N(t)$ for different entrance rates.

3.3. The impact of the three exit rates

By Theorem 1, the I-exit rate affects the equilibrium of the SIPNS model in the following way.

Theorem 3. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is strictly increasing with* δ_I.

(b) I^*, P^* *and* N^* *are strictly decreasing with* δ_I.

Extensive experiments show that, typically, the I-exit rate affects the dynamics of the SIPNS model in the way shown in Fig. 3. In general, it is drawn that, for any I-exit rate, the SIPNS model approaches the equilibrium.

Let

$$\delta_P^* = \alpha_P \sqrt{\frac{\beta_P \delta_P \delta_N}{\alpha_N \beta_N (\alpha_P + \alpha_N + \gamma_I + \delta_I)}} - \gamma_P. \tag{15}$$

By Theorem 1, the P-exit rate affects the equilibrium of the SIPNS model in the following way.

Theorem 4. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is strictly increasing with* δ_P.

(b) P^* *is strictly decreasing with* δ_P.

(c) If $\delta_P^* \le 0$, *then* I^* *and* N^* *are strictly decreasing with* δ_P.

(d) If $\delta_P^* > 0$, *then* I^* *and* N^* *are strictly increasing with* $\delta_P < \delta_P^*$, *are strictly decreasing with* $\delta_P > \delta_P^*$, *and attain their respective maximum at* $\delta_P = \delta_P^*$.

Extensive experiments show that, typically, the P-exit rate affects the dynamics of the SIPNS model in the way shown in Fig. 4. In general, it is drawn that, for any P-exit rate, the SIPNS model approaches the equilibrium.

By Theorem 1, the N-exit rate affects the equilibrium of the SIPNS model in the following way.

Theorem 5. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

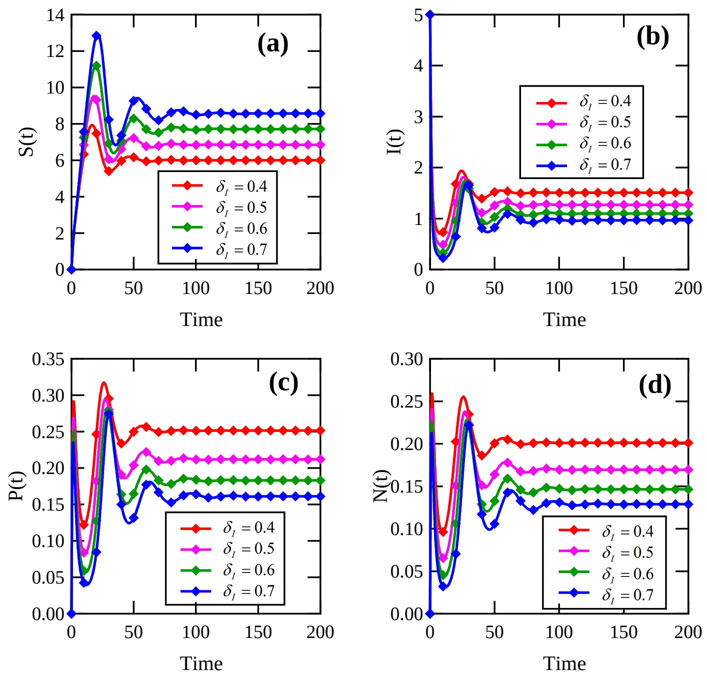

Figure 3. The time plots of $S(t)$, $I(t)$, $P(t)$ and $N(t)$ for different I-exit rates.

(a) *S^* is irrelevant to δ_N.*
(b) *I^* and P^* are strictly increasing with δ_N.*
(c) *N^* is strictly decreasing with δ_N.*

Extensive experiments show that, typically, the N-exit rate affects the dynamics of the SIPNS model in the way shown in Fig. 5. In general, it is drawn that, for any N-exit rate, the SIPNS model approaches the equilibrium.

3.4. The impact of the two comment rates

Let

$$\alpha_P^* = (\gamma_P + \delta_P)\sqrt{\frac{\alpha_N \beta_N(\alpha_N + \gamma_I + \delta_I)}{\beta_P \delta_P \delta_N}}. \tag{16}$$

By Theorem 1, the P-comment rate affects the equilibrium of the SIPNS model in the following way.

Theorem 6. *Consider the equilibrium $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ of the SIPNS model (9). The following claims hold.*

(a) *S^* is strictly decreasing with α_P.*
(b) *P^* is strictly increasing with α_P.*
(c) *I^* and N^* are strictly increasing with $\alpha_P < \alpha_P^*$, are strictly decreasing with $\alpha_P > \alpha_P^*$, and attain their respective maximum at $\alpha_P = \alpha_P^*$.*

Extensive experiments show that, typically, the P-comment rate affects the dynamics of the SIPNS model in the way shown in Fig. 6. In general, it is drawn that, for any P-comment rate, the SIPNS model approaches the equilibrium.

Let

$$\alpha_N^* = \frac{1}{\gamma_P + \delta_P}\sqrt{\frac{\alpha_P \beta_P \delta_I \delta_N(\gamma_P + \delta_P) + \alpha_P^2 \beta_P \delta_P \delta_N}{\beta_N}}. \tag{17}$$

By Theorem 1, the N-comment rate affects the equilibrium of the SIPNS model in the following way.

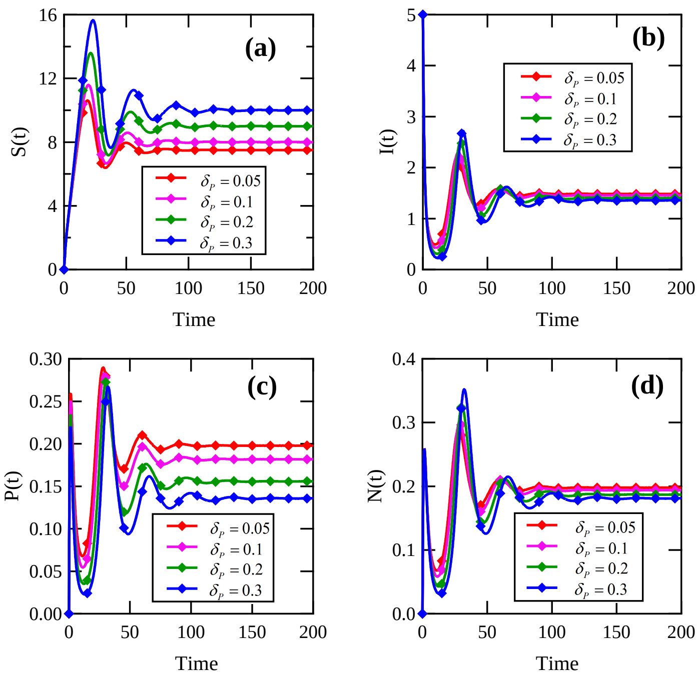

Figure 4. The time plots of $S(t)$, $I(t)$, $P(t)$, and $N(t)$ for different P-exit rates.

Theorem 7. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is strictly increasing with* α_N.
(b) I^* *and* P^* *are strictly decreasing with* α_N.
(c) N^* *is strictly increasing with* $\alpha_N < \alpha_N^*$, *is strictly decreasing with* α_N^*, *and attains the maximum at* $\alpha_N = \alpha_N^*$.

Extensive experiments show that, typically, the N-comment rate affects the dynamics of the SIPNS model in the way shown in Fig. 7. In general, it is drawn that, for any N-comment rate, the SIPNS model approaches the equilibrium.

3.5. The impact of the two infection forces

By Theorem 1, the P-infection rate affects the equilibrium of the SIPNS model in the following way.

Theorem 8. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is strictly decreasing with* β_P.
(b) I^*, P^* *and* N^* *are strictly increasing with* β_P.

Extensive experiments show that, typically, the P-infection force affects the dynamics of the SIPNS model in the way shown in Fig. 8. In general, it is drawn that, for any P-infection force, the SIPNS model approaches the equilibrium.

By Theorem 1, the N-infection rate affects the equilibrium of the SIPNS model in the following way.

Theorem 9. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is irrelevant to* β_N.
(b) I^*, P^* *and* N^* *are strictly decreasing with* β_N.

Extensive experiments show that, typically, the N-infection force affects the dynamics of the SIPNS model in the way shown in Fig. 9. In general, it is drawn that, for any N-infection force, the SIPNS model approaches the equilibrium.

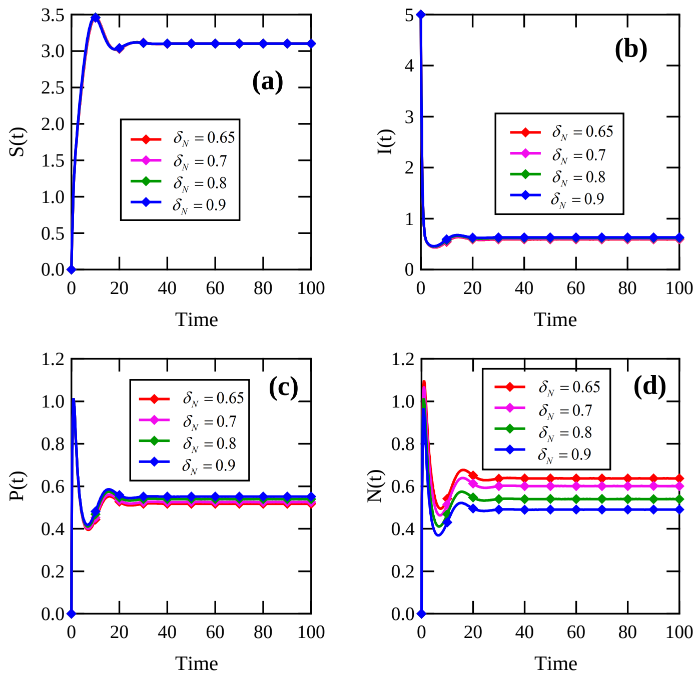

Figure 5. The time plots of $S(t)$, $I(t)$, $P(t)$, and $N(t)$ for different N-exit rates.

3.6. The impact of the two viscosity rates

By Theorem 1, the I-viscosity rate affects the equilibrium of the SIPNS model in the following way.

Theorem 10. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is strictly increasing with* γ_I.
(b) I^*, P^* *and* N^* *are strictly decreasing with* γ_I.

Extensive experiments show that, typically, the I-viscosity rate affects the dynamics of the SIPNS model in the way shown in Fig. 10. In general, it is drawn that, for any I-viscosity rate, the SIPNS model approaches the equilibrium.

By Theorem 1, the P-viscosity rate affects the equilibrium of the SIPNS model in the following way.

Let

$$\gamma_P^* = \alpha_P \sqrt{\frac{\beta_P \delta_P \delta_N}{\alpha_N \beta_N (\alpha_P + \alpha_N + \gamma_I + \delta_I)}} - \delta_P. \tag{18}$$

Theorem 11. *Consider the equilibrium* $\mathbf{E}^* = (S^*, I^*, P^*, N^*)$ *of the SIPNS model (9). The following claims hold.*

(a) S^* *is strictly increasing with* γ_P.
(b) P^* *is strictly decreasing with* γ_P.
(c) If $\gamma_P^* \leq 0$, *then* I^* *and* N^* *are strictly decreasing with* γ_P.
(d) If $\gamma_P^* > 0$, *then* I^* *and* N^* *are strictly increasing with* $\gamma_P < \gamma_P^*$, *are strictly decreasing with* $\gamma_P > \gamma_P^*$, *and attain their respective maximum at* $\gamma_P = \gamma_P^*$.

Extensive experiments show that, typically, the P-viscosity rate affects the dynamics of the SIPNS model in the way shown in Fig. 11. In general, it is drawn that, for any P-viscosity rate, the SIPNS model approaches the equilibrium.

Combining the above discussions, we propose the following conjecture.

Conjecture 1. *The equilibrium of the SIPNS model (9) is globally attracting. That is, starting from any initial state, the model approaches the equilibrium.*

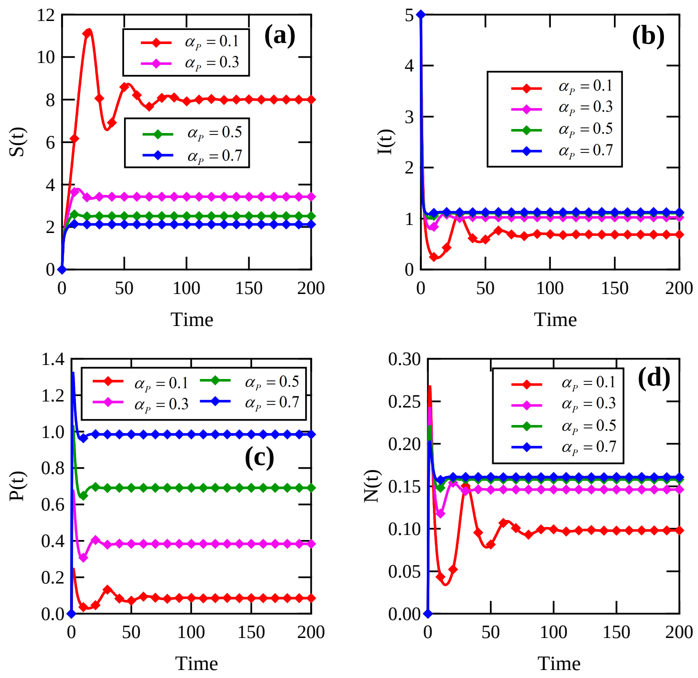

Figure 6. The time plots of $S(t)$, $I(t)$, $P(t)$, and $N(t)$ for different P-comment rates.

4. The expected overall profit of a WOM marketing campaign

This section is dedicated to the study of the impact of different factors on the expected overall profit of a WOM marketing campaign. First, it should be noted that, when T is large enough, the expected overall profit can be approximated by the following quantity.

$$J^* = \beta_P T P^* S^* = T \frac{\mu \alpha_P \beta_P \delta_N (\alpha_P + \alpha_N + \gamma_I + \delta_I)(\gamma_P + \delta_P)}{\alpha_P \beta_P \delta_N (\alpha_N + \delta_I)(\gamma_P + \delta_P) + \alpha_P^2 \beta_P \delta_P \delta_N + \alpha_N \beta_N (\gamma_P + \delta_P)^2 (\alpha_P + \alpha_N + \gamma_I + \delta_I)}. \tag{19}$$

4.1. *The impact of the entrance rate and the three exit rates*

The impact of the entrance rate and the three exits rates on J^* is as follows.

Theorem 12. *Consider the SIPNS model (9). The following claims hold.*

(a) J^* is strictly increasing with μ and δ_N.
(b) J^* is strictly decreasing with δ_I and δ_P.

Extensive experiments show that, typically, the impact of the entrance rate and the three exit rates on the expected overall profit is as shown in Fig. 12. In general, it is drawn that J is strictly increasing with μ and δ_N, and is strictly decreasing with δ_I and δ_P. These findings accord with Theorem 12.

In practice, the entrance rate can be enhanced by launching a viral marketing (VM) campaign. That is, the marketer develops a marketing message and encourages customers to forward this message to their contacts. There are quite a number of successful VM cases: Hotmail generated 12 million subscribers in just 18 months with a marketing budget of only \$50,000, and Unilevers Dove Evolution campaign generated over 2.3 million views in its first 10 days. These VM campaigns were successful in part because the marketers effectively utilized VM's unique potential to reach large numbers of potential customers in a short period of time at a lower cost. The I-exit rate or the P-exit rate can be reduced by enhancing customers' experiences with the purchased products or/and taking promotional measures such as discounting and distributing coupons. Typically, the N-exit rate is uncontrollable.

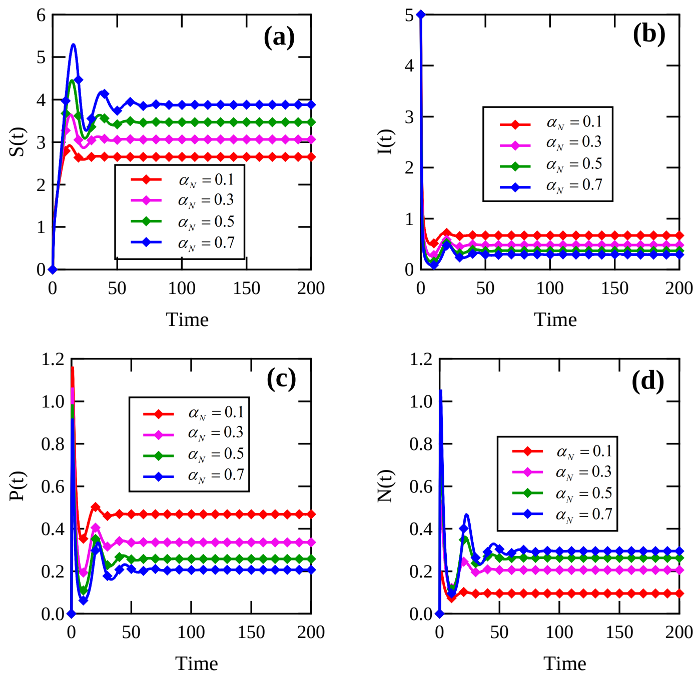

Figure 7. The time plots of $S(t)$, $I(t)$, $P(t)$ and $N(t)$ for different N-comment rates.

4.2. *The impact of the two comment rates*

The impact of the two comment rates on J^* is as follows.

Theorem 13. *Consider the SIPNS model (9). The following claims hold.*

(a) J^* is strictly increasing with α_P.
(b) J^* is strictly decreasing with α_N.

Extensive experiments show that, typically, the impact of the two comment rates on the expected overall profit is as shown in Fig. 13. In general, it is drawn that J is strictly increasing with α_P, and is strictly decreasing with α_N. These findings conform to Theorem 13.

In practice, the P-comment rate can be enhanced, and the N-comment rate can be reduced, by enhancing customers' experiences.

4.3. *The impact of the two infection forces*

The impact of the two infection forces on J^* is as follows.

Theorem 14. *Consider the SIPNS model (9). The following claims hold.*

(a) J^* is strictly increasing with β_P.
(b) J^* is strictly decreasing with β_N.

Extensive experiments show that, typically, the impact of the two infection forces on the expected overall profit is as shown in Fig. 14. In general, it is drawn that J is strictly increasing with β_P, and is strictly decreasing with β_N. These findings agree with Theorem 14. In practice, the P-infection force can be enhanced by enhancing customers' experiences and, hence, earning positive WOM. The N-infection force is often uncontrollable.

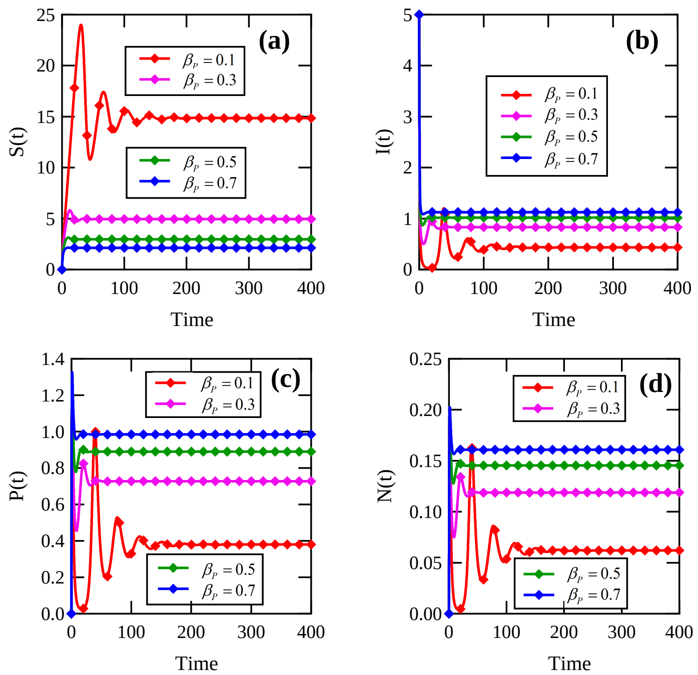

Figure 8. The time plots of $S(t)$, $I(t)$, $P(t)$ and $N(t)$ under different P-infection forces.

4.4. *The impact of the two viscosity rates*

See Eq. (18). The impact of the two viscosity rates on J^* is as follows.

Theorem 15. *Consider the SIPNS model (9). The following claims hold.*

(a) J^* *is strictly increasing with* γ_I.

(b) If $\gamma_P^* \leq 0$, *then* J^* *is strictly decreasing with* γ_P.

(c) If $\gamma_P^* > 0$, *then* J^* *is strictly increasing with* $\gamma_P < \gamma_P^*$, *is strictly decreasing with* $\gamma_P > \gamma_P^*$, *and attains the maximum at* $\gamma_P = \gamma_P^*$.

Extensive experiments show that, typically, the impact of the two viscosity rates on the expected overall profit is as shown in Fig. 15. In general, the following conclusions are drawn.

(a) J is increasing with γ_I.

(b) There is $\gamma_P^{**}(T)$ $(T > 0)$ such that (1) $\gamma_P^{**}(T) \to \gamma_P^*$ as $T \to \infty$, (2) if $\gamma_P^{**}(T) \leq 0$, then J is strictly decreasing with γ_P, and (3) if $\gamma_P^{**}(T) > 0$, then J is strictly increasing with $\gamma_P < \gamma_P^{**}(T)$, is strictly decreasing with $\gamma_P > \gamma_P^{**}(T)$, and attains the maximum at $\gamma_P = \gamma_P^{**}(T)$.

These findings comply with Theorem 15.

In practice, the I-viscosity rate can be enhanced by enhancing customers' experiences or by taking promotional measures.

5. Conclusions and remarks

WOM marketing processes with both positive and negative comments have been modeled as the SIPNS model, and a measure of the overall profit of WOM marketing campaigns has been proposed. The SIPNS model has been shown to admit a unique equilibrium, and the impact of different factors on the equilibrium has been determined. Furthermore, extensive experiments have shown that the equilibrium is much likely to be globally attracting. Finally,

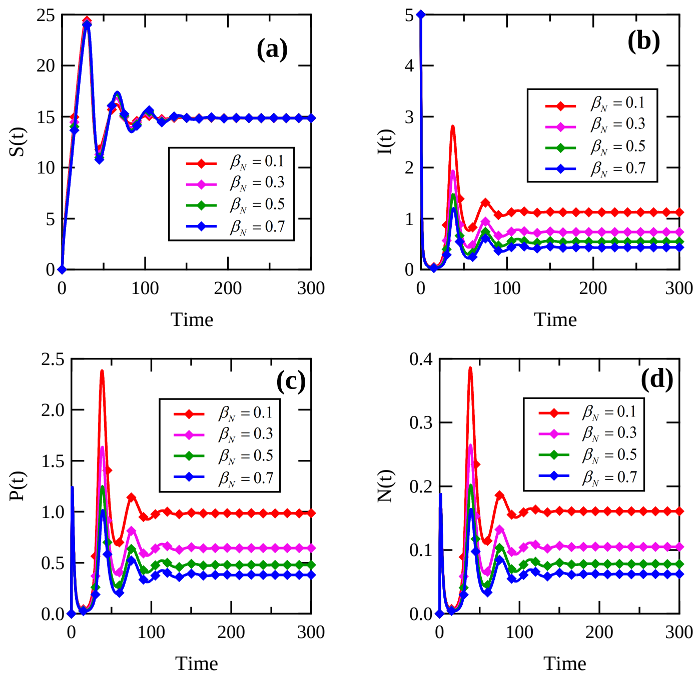

Figure 9. The time plots of $S(t)$, $I(t)$, $P(t)$ and $N(t)$ under different N-infection forces.

the influence of different factors on the expected overall profit has been ascertained. On this basis, some promotion strategies have been suggested.

Toward this direction, lots of efforts are yet to be made. It is well known that the structure of the WOM network has significant influence on the performance of a viral marketing [35, 36]. The proposed SIPNS model is a population-level model and hence does not allow the analysis of this influence. To reveal the impact of the WOM network on the performance, network-level spreading models [37–40] or individual-level spreading models [41–47] are better options. The profit model presented in this paper builds on the uniform-profit assumption. However, in everyday life different products may have separate profits. Hence, it is of practical importance to construct a non-uniform profit model. Also, a customer may purchase multiple items a time, and the corresponding model is yet to be developed. Typically, WOM marketing campaigns are subject to limited budgets. The dynamic optimal control strategy against malicious epidemics [48–52] may be borrowed to the analysis of WOM marketing so as to achieve the maximum possible net profit.

Acknowledgments

The authors are grateful to the anonymous reviewers and the editor for their valuable comments and suggestions. This work is supported by Natural Science Foundation of China (Grant No. 61572006), National Sci-Tech Support Program of China (Grant No. 2015BAF05B03), and Fundamental Research Funds for the Central Universities (Grant No. 106112014CDJZR008823).

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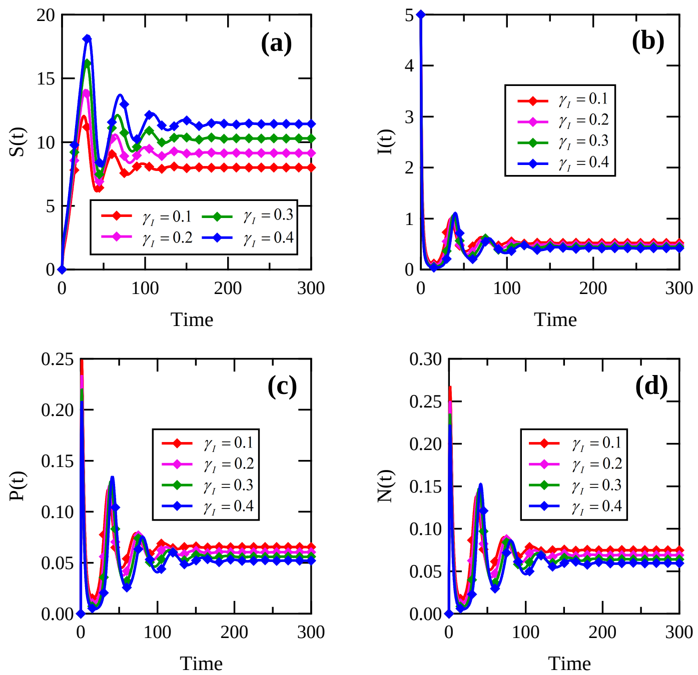

Figure 10. The time plots of $S(t)$, $I(t)$, $P(t)$, and $N(t)$ for different I-viscosity rates.

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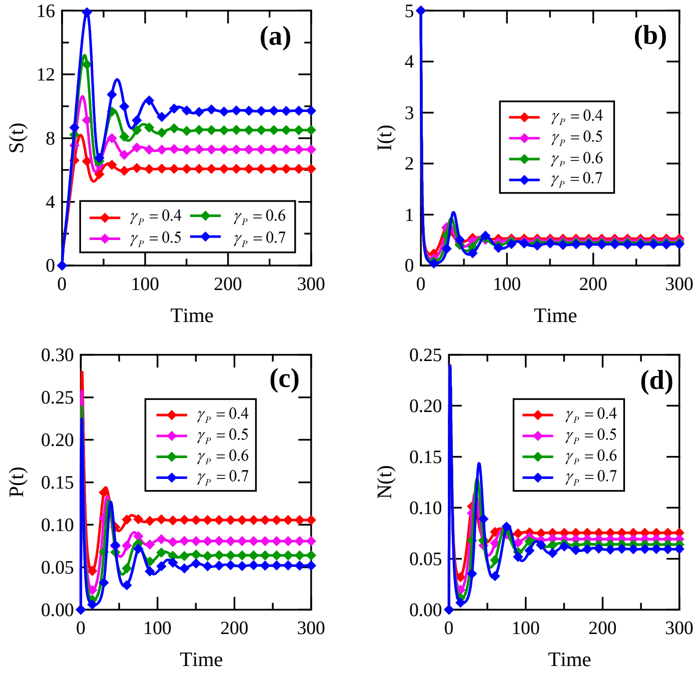

Figure 11. The time plots of $S(t)$, $I(t)$, $P(t)$, and $N(t)$ for different P-viscosity rates.

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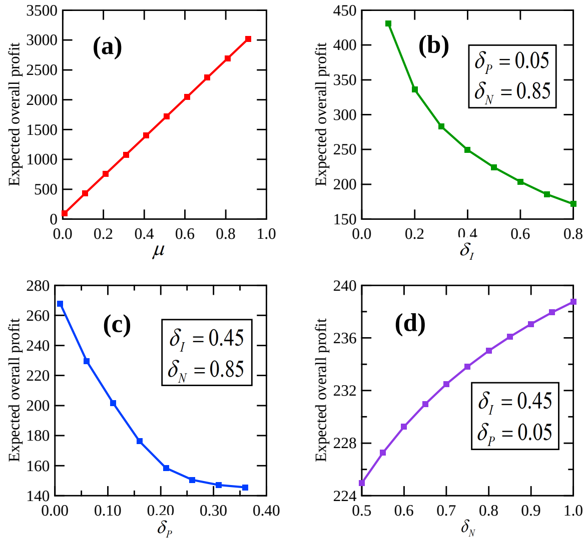

Figure 12. The expected overall profit versus (a) the entrance rate, (b) the I-exit rate, (c) the P-exit rate, and (d) the N-exit rate.

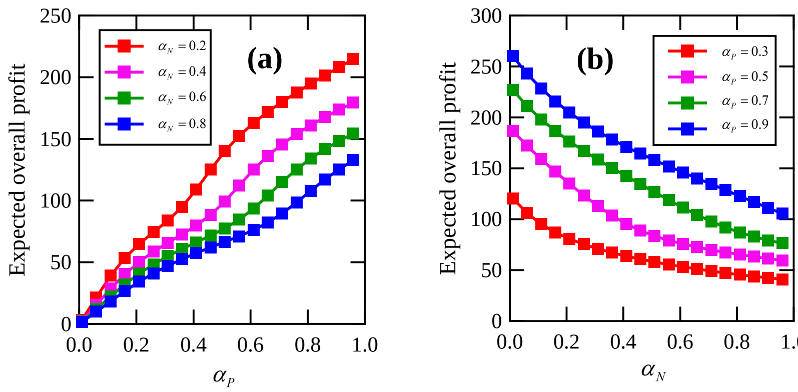

Figure 13. The expected overall profit versus (a) the P-comment rate, and (b) the N-comment rate.

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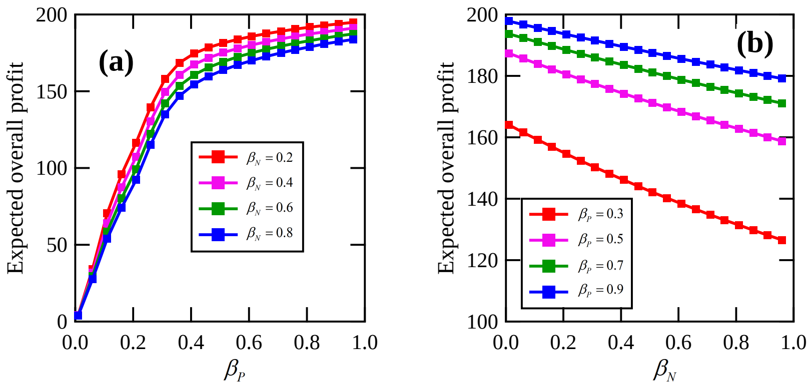

Figure 14. The expected overall profit versus (a) the P-infection force, and (b) the N-infection force.

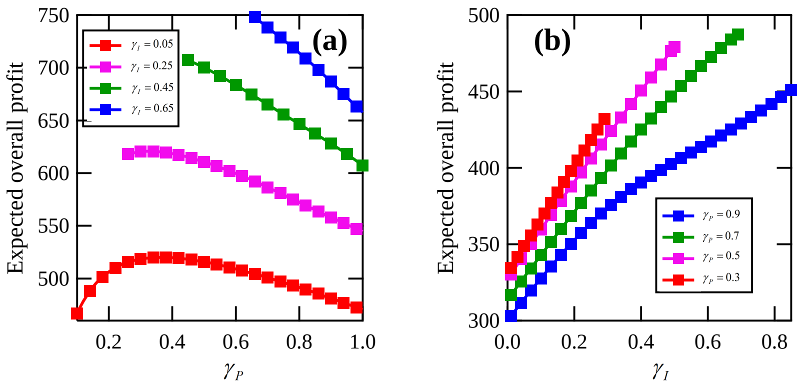

Figure 15. The expected overall profit versus (a) the P-viscosity rate, and (b) the I-viscosity rate.