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Fractional matching markets ☆

Vikram Manjunath

Department of Economics, Texas A&M University, United States

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A B S T R A C T

I take a decentralized approach to fractional matching with and without money. For the model with money I define and show the existence of competitive equilibria. For the model without money, while competitive equilibria may not exist, I define a version of approximate equilibrium and show existence.

From a welfare standpoint, I show that equilibrium allocations are in the core. Similarly, approximate equilibrium allocations are in the approximate version of the core.

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1. Introduction

In the traditional marriage model, each agent is matched to a single partner once and for all. Many-to-many models, on the other hand, allow each agent to be matched to several partners. Between these two models lie problems where each agent can be matched to several partners, but only to one at a given time or to a probability distribution over partners. These are *fractional matching* problems.

For an instance of a fractional matching problem one can think of a law firm with many partners and associates. Each partner cares about which associate works for him and each associate cares about which partner he works for. Yet an associate can split his time between working for different partners. Or one can think of a surgical residency program with several attending physicians and residents. A resident cares about whom he assists while an attending physician cares about who assists her. Another example is a market where traders care about the identities of their trading partners.

Among all of the rationales for considering fractional matching, two stand out. The first is fairness: the restriction to discrete matches precludes treating two, otherwise identical, agents equally. The second is efficiency: shoehorning a problem into a discrete model leads to inefficiency. To illustrate this, think of the following very unhappy Gale–Shapley world involving two boys Andy and Bob and two girls Yvonne and Zelda at a dance. Andy would like to dance with Yvonne. Unfortunately, Yvonne likes Bob and would prefer to dance with him. Bob, however, is hoping to dance with Zelda who, in turn, wants to dance with Andy. To make matters worse, suppose that each of them would rather not dance than dance only with the less preferred partner. Under the restriction that a pair can dance together the whole night or not at all, the

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E-mail address: vmanjunath@tamu.edu.

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only reasonable thing to do is for everyone to go home without dancing at all. Yet, it could be that each of these boys and girls would rather spend half the evening dancing with each partner than not dance at all. If they are willing to accept *fractional matching*, we could arrange it so that Andy dances with Yvonne and Bob dances with Zelda for the first half of the evening and for the second half Andy dances with Zelda and Bob dances with Yvonne.

The focus of this paper is to define and show existence of competitive equilibria, thereby establishing that these problems can reasonably be dealt with as decentralized markets.

I start with a version of the model where agents are not only matched, but also consume a tradable good that one might call "money."[1] For such economies, I define and show existence of competitive equilibria (Theorem 1). A key insight to defining an equilibrium is that the goods that agents are trading (other than money) are not private goods. These equilibria involve price systems that are "double indexed" so that two different agents need not necessarily pay the same price to be matched with the same partner. Thus, I refer to these equilibria as *double-indexed price* (DIP) equilibria. The proof of existence involves some intricacies to handle the lack of free disposal: in our earlier example, Zelda cannot "discard" the time that Andy spends dancing with her.

Next, I turn to a model without money. That is, each agent consumes only a fractional matching and is endowed with himself but no money. The notion of a DIP equilibrium is still meaningful here. The only difference is that agents neither value nor are endowed with money, so the money terms drop out from either side of each agent's budget constraint. Unfortunately, I show a DIP equilibrium may not exist for such economies. One reason for nonexistence is that some agents may not have positive wealth at given prices. A seemingly straightforward solution to this problem is to simply endow each agent with a positive amount of fake money as done by Hylland and Zeckhauser (1979) and then allow them to trade just as in the case with money. The trouble is that requiring every agent to have positive wealth is not enough.[2] Consider, for instance, a different situation involving Andy and Yvonne. There is nothing Andy would like more than to dance with Yvonne. However, Yvonne would prefer not to dance at all. No matter the prices or the endowments of fake money, she can afford to consume her time endowment by not dancing at all, which is also what she would choose. On the other hand, at every finite price, Andy's demand for her time is always positive if he has a positive endowment of fake money. Thus, no finite price can clear this market.[3]

I propose a different solution. Since there is no natural numeraire, suppose that prices are quoted in terms of some fake money.[4] However, no agent is actually endowed with any of this money. So at a DIP equilibrium, each agent's net expenditure of this fake money is non-positive. Thus, aggregate expenditure of money is non-positive and no external money needs to be injected into the economy. Since an equilibrium may not exist, suppose that the market maker can be tasked with imposing "small" redistributions of fake money. That is, some agents are given a small deficit and others are given a small surplus on the right hand side of their budget constraint. The smallness of these redistributions are in terms of welfare: an agent who is given a deficit can object to the market maker's decision if it reduces his utility by more than ε. In equilibrium, the market maker settles on a redistribution that is immune to such challenges and prices are such that each agent maximizes his preferences subject to his budget constraint under this redistribution. I show that such εDIP equilibria exist (Theorem 2).[5]

From a welfare perspective, when a DIP equilibrium exists, I show that it is in the core (Proposition 1). For an economy without money, however, the core may be empty. Nonetheless, an εDIP equilibrium is both Pareto efficient (Proposition 2) and in the ε-core (Proposition 3).

The decentralized approach to fractional matching that I present here is novel. Previous work on fractional matching deals with centralized mechanisms. Furthermore the main concern is "ex post stability" (Rothblum, 1992; Roth et al., 1993; Aldershof et al., 1999; Baïou and Balinski, 2000; Klaus and Klijn, 2006; Sethuraman et al., 2006). There are some exceptions. Bogomolnaia and Moulin (2004) consider the *ex ante* properties of probabilistic matches in a one to one setting when preferences are such that there are exactly two indifference classes of mates: acceptable and unacceptable. Manjunath (2013) and Doğan and Yıldı z (2016) sort out various notions of the *ex ante core* and *ex ante pairwise stability* for random matches. Gudmundsson (2015) looks into strategy-proof rules that select such allocations. Kesten and Utku Ünver (2015) point out that school choice problems are typically probabilistic, and hence fractional, matching problems. This is because school priorities are coarse and ties are broken randomly. They study *ordinal ex ante* stability concepts for these problems. For a general class of fractional matching problems, Andersson et al. (2014), Chiappori et al. (2014), and Alkan and Tuncay (2014) show that while, in general, the core may be empty when restricted to all or nothing matches, when half-matches are allowed the core is nonempty.

Baïou and Balinski (2002) and Alkan and Gale (2003) also study the scheduling problem. Like other work on fractional matching, their goal is to find a stable schedule in the sense that the scheduled assignment is stable at each point in time.

[1] This is still a non-transferable utility model unlike Shapley and Shubik (1969).

[2] See Appendix D for an example of an economy that does not have a DIP equilibrium when preferences are only *weakly* monotonic in money, as would be the case with fake money.

[3] Even in the setting of Hylland and Zeckhauser (1979), if we start from an endowment and then allow agents to trade probability shares that they own, an equilibrium may not exist.

[4] Since the price of a given good may be zero, it may not be possible to simply quote relative prices.

[5] Whereas DIP equilibrium relies only on ordinal information about preferences, εDIP equilibrium is a cardinal notion.

Stability and strategy-proofness have long been central in the literature on matching. However, as with classical exchange economies (Hurwicz, 1972; Postlewaite, 1985; Barberà and Jackson, 1995; Schummer, 1996; Serizawa, 2002) and economies with public goods (Schummer, 1999), it is easy to show that Pareto-efficiency, strategy-proofness, and individual rationality are incompatible in the fractional matching model. Moreover, the informational content of a preference profile is very large. So even if these properties were compatible, it would be unreasonable to expect that participants could transmit all relevant information so that a central authority might apply such a rule. The *competitive solution* has been central in addressing similar concerns for exchange economies. It is both informationally efficient (Hurwicz, 1977; Osana, 1978; Jordan, 1982) as well as decentralizeable (Sonnenschein, 1974).[6]

The analysis in this paper highlights the fact that pairs in a matching model are a sort of club good (Buchanan, 1965). As in economies with public goods (Lindahl, 1958), club goods rule out anonymous prices. Cole and Prescott (1997) study economies with club goods and a continuum of traders. They assume that each person owns some amount of private goods that are exchangeable and are used to produce club goods. They show existence of competitive equilibria assuming a finite number of types but a continuum of agents of each type. More closely related to the current paper is the work of Ellickson et al. (1999, 2001). They assume that club membership is indivisible, but each person can be a member of up to an exogenously specified number of clubs. For large economies (either a continuum or a finite but large number of agents) with money, they show the existence of equilibria that are similar to the ones defined here (approximate equilibria for the finite case).[7] Contrary to the setting that I study here, competitive equilibria and core allocations do not exist for finite economies in their setting. The main contribution of this paper to the literature on club goods is to show that by considering fractional membership, earlier results that have been obtained only for limit economies hold for finite economies.

For expositional clarity, I have kept the model simple. However, as I explain in Section 6, all of the analysis readily generalizes in several ways: agents could be partitioned according to their attributes in order to avoid personalized prices to model large markets; the bipartite structure that governs which agents can be matched could be dropped to accommodate arbitrary graphs that model social networks; even the assumption that agents are matched to form only pairs may be relaxed to study fractional coalition formation.

The remainder of the paper is organized as follows. I introduce the model in Section 2. In Section 3, I define DIP equilibria and show their existence when every agent is endowed with money. For economies without money, I define and show the existence of εDIP equilibria in Section 4. I show that equilibrium allocations are in the appropriately defined core in Section 5. I discuss extensions of the model in Section 6. All proofs are in the Appendix.

2. The model

A *fractional matching problem* consists of a set of agents N partitioned into two sets M and W.

For simplicity, suppose that each agent has unit availability.[8] For each $m \in M$, a *fractional partnership allocation* divides his time between partners in W and being alone.[9] That is, a fractional partnership allocation for m is

$$\pi_m \in \Delta_m \equiv \left\{ y_m \in \mathbb{R}_+^{W \cup \{m\}} : \sum_{i \in W \cup \{m\}} y_{mi} = 1 \right\}.$$

For each $w \in W$, π_{mw} represents the time that m spends with w while π_{mm} is the time that he spends alone.

For each $w \in W$, define Δ_w similarly and let $\Delta \equiv \underset{i \in N}{\times} \Delta_i$,

Consider a profile of fractional partnership allocations, $\pi \in \Delta$. For each pair $m \in M$ and $w \in W$, π_{mw} is the time that m spends with w and π_{wm} is the time that w spends with m. For this to make any sense, these must be the same: the time that they spend together. So feasibility requires that $\pi_{mw} = \pi_{wm}$. Let Π be the set of *feasible fractional partnership allocations*.

Aside from spending time with others, agents also consume a divisible private good that I call money, but depending on the application could be any divisible good.[10] Each $i \in N$ is endowed with $\omega_i \in \mathbb{R}_+$ units of money. Since i consumes a fractional partnership allocation along with an amount of money, his consumption set is $X_i \equiv \Delta_i \times \mathbb{R}_+$. Let $X \equiv \underset{i \in N}{\times} X_i$.

At money-endowments $\omega \in \mathbb{R}_+^N$, a feasible allocation is $(\pi, x) \in X$ such that π is feasible and x allocates all of the money in the economy. That is, $\pi \in \Pi$ and $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$. Let $Z(\omega)$ be the set of feasible allocations.

[6] Hylland and Zeckhauser (1979) showed that a competitive approach is fruitful for the probabilistic assignment of objects to agents. Subsequently, Budish (2011) has extend this to "combinatorial assignment" problems. In the latter case, however, Budish (2011) uses an approximate equilibrium notion. The approximateness in his definition of equilibrium is in terms of feasibility as opposed to the budget constraint. He et al. (2015) use this approach in a school choice setting.

[7] Zame (2007) and Scotchmer and Shannon (2015) study similar economies with the difference that individual types are private information, thereby allowing them to analyze issues like moral hazard, adverse selection, signaling, and insurance.

[8] This restriction is consistent with the interpretation of fractional allocations as probability distributions. However, all of the analysis is robust to allowing each agent to have a different availability.

[9] For the sake of exposition, I will refer to this as time. However, this is an abstract model that can be applied to many situations. For instance, if one were interested in probabilistic matching then this would be a probability distribution over partners.

[10] The assumption that there is a single private good is not essential. The analysis generalizes in a straightforward manner for any number of such goods.

For each $i \in N$, i's preferences over his consumption set are described by a utility function $u_i : X_i \to \mathbb{R}$. I assume that u_i is continuous and quasi-concave.[11] I also assume that it is monotonic in money. That is, for each $\pi_i \in \Delta_i$ and each pair $x_i, x_i' \in \mathbb{R}_+$, if $x_i > x_i'$ then $u_i(\pi_i, x_i) > u_i(\pi_i, x_i')$. Let $\mathcal{U}_i$ be the set of such utility functions. Leaving M and W fixed, an economy is fully described by a profile of utility functions $u \in \mathcal{U} \equiv \times_{i \in N} \mathcal{U}_i$ and a profile of money-endowments $\omega \in \mathbb{R}_+^N$. Let $\mathcal{E} \equiv \mathcal{U} \times \mathbb{R}_+^N$ be the set of economies.

3. Economies with money

In an exchange economy, the market clearing condition requires aggregate demand for each good to equal supply. That amounts to requiring, in the current model, the aggregate "demand" for each pair to equal its "supply." That is, the clearing condition requires a coincidence of demands among the members of a pair. Since a consumer is also "consumed," a single price for each agent's time is not enough to adequately reflect both sides' preferences. One way to think about this is as follows: If m prefers w_1 to w_2, he would not charge them the same price. He would be inclined to charge w_1, whom he prefers, less than he charges w_2. Thus for each pair $i, j \in N$, p_{ij} is the price that i pays for a unit of j. I call these "double-indexed" prices.

As explained in Section 2, π_{ij} and π_{ji} represent the same thing: the time that i and j spend together. However, both p_{ij} and p_{ji} are prices for this good: p_{ij} is the price from i's point of view and p_{ji} is the price from j's point of view. These prices only make sense if $p_{ij} = -p_{ji}$. If p_{ij} is negative, it means that i receives money for spending time with j.[12] That is, j is paying for i's time. The reverse is true if p_{ij} is positive.[13] Since i owns all his own time, no matter what he pays for solitude, he pays himself. Thus, the effective price of i's time to himself, p_{ii}, is zero.

Finally, since money is a private good that all agents consume, it has a common price p_0. While I could normalize p_0, I leave it this way to emphasize that prices need not be quoted in terms of money, particularly since I will consider the case without money in Section 4.

Thus, for each $m \in M$, the price vector that m faces is $p_m \equiv (0, (p_{mw})_{w \in W}, p_0)$. Each $w \in W$ faces a similarly defined p_w. Let $\mathbb{P}$ be the set of all such price systems.[14]

Budget sets Each $i \in N$ starts out owning all of his time as well as some money. The value of his time to himself is zero since the price of his time for himself is zero. The value of his endowment of money is $\omega_i p_0$. Thus, the right-hand side of his budget constraint is $\omega_i p_0$. On the other hand if he consumes the bundle (π_i, x_i), it costs him $(\pi_i, x_i) \cdot p_i$. Thus, facing prices p_i, he must choose from the following budget set:

$$B_i(p, \omega_i) \equiv \{(\pi_i, x_i) \in X_i : (\pi_i, x_i) \cdot p_i \le \omega_i p_0\}$$

Competitive equilibrium Much like the usual definition of competitive equilibrium, I define an equilibrium as an allocation and a price system where every agent is maximizing his utility subject to his budget constraint and the allocation is feasible. Formally, given a $(u, \omega) \in \mathcal{E}$, a *double-indexed price (DIP) equilibrium* is $((\pi, x), p) \in X \times \mathbb{P}$ such that for each $i \in N$, $(\pi_i, x_i) \in \operatorname{argmax}_{B_i(p,\omega_i)} u_i$ and $(\pi, x) \in Z(\omega)$.

My first main result is that as long as each agent is endowed with a positive amount of money, a DIP equilibrium exists.

Theorem 1. *Every economy where each agent is endowed with a positive amount of money has a DIP equilibrium.*

4. Economies without money

Theorem 1 only covers cases where each person is endowed with a strictly positive amount of money. When money is available and permitted in the economy, such a "survival assumption" is reasonable. Yet, in many interesting applications the only goods available are pairs as money is ruled out.[15]

In this section, I consider a special case of the general model in which agents are neither endowed with nor consume money. As I show below (Example 1 and Appendix D), DIP equilibrium is not the appropriate solution concept for such economies since Theorem 1 only applies if each agent has a positive endowment of money *and* has strictly monotonic preferences for it. Consequently, I define an alternate solution concept that approximates DIP equilibrium in a very natural way. Since the approximation is in welfare terms, the interpretation of the model is slightly different as utilities are cardinal rather than ordinal.

[11] Linear (von Neumann–Morgenstern) preferences would do for probabilistic matching applications. However, for other applications, a richer class of preferences may be appropriate.

[12] As in the current context, prices for club goods may need to be negative (see Allouch and Wooders, 2008 for yet another instance of this).

[13] Another way of thinking about this is that total expenditure by i and j on their time together should be zero.

[14] Let $p \in \mathbb{P}$. For each $m \in M$ and each $w \in W$, $p_{mm} = 0$, $p_{ww} = 0$, and $p_{mw} = -p_{wm}$. So p can be identified by an element of $\mathbb{R}^{M \times W \cup \{0\}}$.

[15] See Roth (2007) for an explanation of how "repugnance" is a constraint for real world market design that rules out money for certain problems.

Since no agent values money, to simplify notation, I drop money and restrict the consumption set of each $i \in N$ to Δ_i. Thus, his preferences are represented by a utility function $u_i : \Delta_i \to \mathbb{R}$. That is, I take preferences only over partnership allocations as the primitive here, rather than over money as well. Since I define an approximate equilibrium concept in what follows, which is cardinal, I restrict attention to continuous utility functions that are concave rather than just quasi-concave. Let $\hat{\mathcal{U}}_i$ be the set of such utility functions. An economy is now just a profile of utility functions. Let $\hat{\mathcal{U}} \equiv \underset{i \in N}{\times} \hat{\mathcal{U}}_i$ be the set of economies without money.

Since there is no money in the economy, the set of feasible allocations is just Π. Let $\hat{\mathbb{P}}$ be the set of price systems in $\mathbb{P}$ without a price for money. So, given $p \in \hat{\mathbb{P}}$, each $m \in M$ faces the price vector $p_m \equiv (0, (p_{mw})_{w \in W})$. Each $w \in W$ faces a similarly defined p_w.

Budget sets Without any endowment of money, all that $i \in N$ owns at the outset is his own time. This has a value of zero. Thus, the right-hand side of his budget constraint is zero. The left-hand side, however, is just the value of his partnership allocation. Thus, facing prices p_i, he must choose from the following budget set:

$$B_i(p) \equiv \{\pi_i \in \Delta_i : \pi_i \cdot p_i \le 0\}.$$

Competitive equilibrium Just as before, an equilibrium is an allocation and a price system where every agent maximizes his utility subject to his budget constraint and the allocation is feasible. Formally, given $u \in \hat{\mathcal{U}}$, a *DIP equilibrium* is $(\pi, p) \in \Delta \times \hat{\mathbb{P}}$ such that for each $i \in N$, $\pi_i \in \underset{B_i(p)}{\operatorname{argmax}}\, u_i$ and $\pi \in \Pi$.

Unfortunately, as Example 1 demonstrates, double-indexed prices are not enough to guarantee the existence of equilibria. There are two problems. The first is that each agent's endowment is on the boundary of his consumption set: he is endowed with his own time and nothing else. Thus, he might not have positive wealth at certain prices. The second is that preferences are necessarily satiated since each agent's consumption set is compact.

Example 1 *(An economy with no DIP equilibria).* There are two agents in N, m and w. While m prefers to spend as much time with w as possible, w would ideally spend only half of the available time with m. This would be the case if $u \in \hat{\mathcal{U}}$ were defined, for instance, as follows: For each $\pi_m \in \Delta_m, u_m(\pi_m) = \pi_{mw}$ and for each $\pi_w \in \Delta_w, u_w(\pi_w) = -(\frac{1}{2} - \pi_{wm})^2$.

Suppose that this economy has a DIP equilibrium (π, p).

Consider the price that w pays for m, p_{wm}. First, it cannot be zero: Otherwise, both agents face zero prices and maximize their preferences without a budget constraint. Thus, m spends all of the time with w while w spends only half of the time with m, which is not feasible.

Second, it cannot be positive: If it is, then w has no income and can spend no time with m. However, m faces a negative price and chooses to spend all of his time with w. Again, this is not feasible.

Third, it cannot be negative: If it is, then m faces a positive price but has no income. Thus, m cannot spend any time with w. However, w chooses to spend half the time with m, which is also infeasible.

Thus, there are no DIP equilibria.

In Example 1, the reason that a DIP equilibrium does not exist is that m does not have positive wealth when p_{mw} is positive. If I followed the approach of Hylland and Zeckhauser (1979) and endowed each agent with a positive amount of fake money and required money to have positive value in equilibrium, then such an equilibrium would exist for the economy defined in Example 1. However, the assumption that agents' preferences are *strictly* monotonic in the amount of money that they consume, which is not the case for fake money, is critical in proving Theorem 1. Appendix D contains an example that demonstrates this. The main insight is that when each agent has a positive amount of fake money, he is able to demand a positive amount of another agent's endowment for every finite price, even if she prefers to consume her own endowment entirely. Such a market does not clear at any price.

As discussed in the introduction, my approach is similar to that of Hylland and Zeckhauser (1979) in that prices are quoted in terms of fake money. However, agents are not endowed with fixed positive amounts of this money. The aggregate endowment of such money in the economy is zero. If no redistribution is made between the agents, then each agent's budget set consists of those bundles that can be purchased with no net expenditure. As shown above, such equilibria may not exist. Suppose that the market maker is able to impose redistributions of the zero aggregate endowment of fake money so that some agents face deficits while others face surpluses. The market maker is, however, constrained by the ability of an agent to object to a deficit imposed on him. If the imposed deficit causes more than an ε loss of utility, then the agent objects. In equilibrium, the market maker settles on a redistribution that no agent objects to.

Below I define budget sets with transfers and a version of DIP equilibrium with redistributions as described above.[16]

[16] These are like "dividend equilibria" of Aumann and Drèze (1986) or "competitive equilibria with slack" of Mas-Colell (1992) where the dividends or slacks sum to 0.

Budget sets with transfers The right-hand side of the budget constraint is now the transfer to the agent while the left-hand side is just as before.

Given an arbitrary transfer t_i and facing prices p_i, i's "budget set with transfer t_i" is

$$\hat{B}_i(p, t_i) \equiv \{\pi_i \in \Delta_i : \pi_i \cdot p_i \leq t_i\}.$$

Approximate competitive equilibria An approximate equilibrium consists of not only an allocation and a price system, but a vector of transfers as well. Since the transfers are a redistribution of wealth, they sum to zero. Each agent maximizes his utility in his budget set given his transfer. As explained above an additional requirement for this to constitute an approximate equilibrium is that these transfers do not cause "large" welfare losses to those who are negatively affected.

Formally, given $u \in \hat{\mathcal{U}}$ and $\varepsilon > 0$, an *εDIP equilibrium* is $(\pi, p, t) \in \Delta \times \hat{\mathbb{P}} \times \mathbb{R}^N$ such that for each $i \in N$,

1. $\pi_i \in \underset{\hat{B}_i(p,t_i)}{\operatorname{argmax}}\ u_i$,
2. $\pi_i \cdot p_i = t_i$, and
3. for each $\pi_i' \in \Delta_i$, if $\pi_i' \cdot p_i \leq 0$ then $u_i(\pi_i) \geq u_i(\pi_i') - \varepsilon$,

and also $\pi \in \Pi$ and $\sum_{i \in N} t_i = 0$.

Recall that for each $u \in \hat{\mathcal{U}}$ and each $i \in N$, I have only assumed that u_i is concave. Without strict concavity, i may have more than one maximizer in his budget set. Think of the extreme case where u_i is a constant function: every bundle is a maximizer. To show, in Section 5, that equilibrium allocations are efficient (without excluding such preferences), an additional condition needs to be met: Each agent must not only maximize in his budget set, but must do so as cheaply as possible. An εDIP equilibrium (π, p, t) is *cost minimizing* if for each $i \in N$ and each $\pi_i' \in \underset{\hat{B}_i(p,t_i)}{\operatorname{argmax}}\ u_i$ it holds that $\pi_i \cdot p_i \leq \pi_i' \cdot p_i$.[17]

The next result says that a cost minimizing εDIP equilibrium always exists (Theorem 2). The proof suggests another interpretation of εDIP equilibria. In order to prove Theorem 2, I define an auxiliary economy. This economy is identical to the original economy except for the introduction of ε units of money. By Theorem 1, as long as each person is endowed with a positive (no matter how small) amount money, the auxiliary economy has an *exact* equilibrium. For very small endowments of the additional good, the auxiliary economy is an approximation of the original economy. An equilibrium of this approximate economy is therefore an approximate equilibrium of the original economy.

Theorem 2. *For each $\varepsilon > 0$, every economy without money has a cost minimizing εDIP equilibrium.*

Recall that the economy described in Example 1 did not have a DIP equilibrium. It is easy to check that for any $\varepsilon > 0$, the tuple (π, p, t) such that $p_{mw} = 2\varepsilon$, $\pi_{mw} = \frac{1}{2}$, $t_m = \varepsilon$, and $t_w = -\varepsilon$ describes an εDIP equilibrium.

5. Welfare

One of the main normative criteria for allocations in the literature on discrete matching problems is *stability* which turns out to be equivalent to the *core*. For the current model, there are two ways to define the core, each based on a different notion of blocking. I define these below.

The core is the set of allocations that no group of agents can "block." That is, no group of agents can match among themselves in a way that makes them better off. Formally, $(\pi, x) \in Z(\omega)$ is *blocked* by $N' \subseteq N$ if there is $(\pi', x') \in Z(\omega)$ such that

1. Members of N' are only matched with one another. That is, for each $i \in N', \pi_{iN'} = 1$.[18]
2. Members of N' redistribute their money-endowments among themselves. That is, $\sum_{i \in N'} x_i' = \sum_{i \in N'} \omega_i$.
3. Every member of N' is at least as well of at (π', x') as at (π, x) and at least one member is better off.

It is *strongly blocked* by N' if there is $(\pi_i', x_i') \in Z(\omega)$ that satisfies conditions 1. and 2. above and

3′. Every member of N' is better off at (π', x') than at (π, x).

[17] Hylland and Zeckhauser (1979) impose this condition in their definition of an equilibrium.

[18] For each $i \in N, \pi_i \in \Delta_i$, and $N' \subseteq N$, I write $\pi_{iN'}$ to denote $\sum_{j \in N'} \pi_{ij}$.

If (π, x) is not blocked by any group, it is in the strong core. If it is not strongly blocked by any group, it is in the weak core.

I first show that DIP equilibrium allocations are in the strong core. Incidentally, this establishes that the strong core is nonempty for economies with strictly positive money-endowments.[19]

Proposition 1. *For every economy with money, each DIP equilibrium allocation is in the strong core.*

In contrast, an economy without money may not have any strong core allocations.[20] Thus, one cannot expect an εDIP equilibrium allocation, π, to be in the core. At a minimum, though, π should be Pareto-efficient: there ought to be no way to make an agent better off without making another agent worse off. That is, there should not be $\pi' \in \Pi$ such that for each $i \in N$, $u_i(\pi'_i) \geq u_i(\pi_i)$ and for some $i \in N$, $u_i(\pi'_i) > u_i(\pi_i)$.

While not every εDIP equilibrium allocation is Pareto-efficient,[21] every *cost minimizing* εDIP equilibrium allocation is.[22]

Proposition 2. *For every $\varepsilon > 0$ and every economy without money, each cost minimizing εDIP equilibrium allocation is Pareto-efficient.*

Though the core may be empty, it is straightforward to apply Theorem 1 of Scarf (1967) to establish that the weak core is nonempty. While εDIP equilibrium allocations may not be in the weak core, they are close for small ε. I define the ε-core and show that εDIP equilibrium allocations are in it (Proposition 3).

For each $u \in \hat{\mathcal{U}}$ and $\pi \in \Pi$, a group $N' \subseteq N$ *ε-strongly blocks π at u* if the members of N' can match among themselves in a way that gives each of its members a utility higher than at π by at least ε. That is, there exists $\pi' \in \Pi$ such that for each $i \in N'$, $\pi'_{iN'} = 1$ and $u_i(\pi'_i) > u_i(\pi_i) + \varepsilon$. If $\pi \in \Pi$ is not ε-strongly bocked by any group at u, then it is in the *ε-core* at u.

Proposition 3. *For every $\varepsilon > 0$ and every economy without money, each εDIP equilibrium allocation is in the ε-core.*

6. Discussion

I have kept the model in this paper simple in the interest of clarity. The analysis, however, is robust to several extensions. A general model that includes the following features along with a generalization of the definition of DIP equilibrium and a proof of existence are in Appendix E.

Large markets and fairness As with Lindahl equilibria (Lindahl, 1958), personalized prices undermine the assumption that agents are price-takers. However, the model that I have presented here can be extended in the following way: each agent is associated with a "kind" which describes his *external characteristics* (as opposed to a "type" which would also encapsulate *internal characteristics* such as the agent's preferences over others). That is, his kind determines how other agents feel about him, but not necessarily how he feels about them. For instance, for a labor market application, the workers could be partitioned into various kinds based on their training, tenure, achievements, and so on, while managers could be partitioned based on the kind of work they are asking workers to do.[23]

In such a model, prices need only be indexed by kinds. Rather than saying what agent i pays to be with agent j, the prices would specify what an agent of kind k pays to be with an agent of kind l. Thus, every agent of the same kind faces the same prices.

Beyond justifying the price-taking assumption, this also ensures that agents are treated "fairly" in the sense that agents of the same kind face the same prices. In classical exchange economies, given private endowments, a competitive equilibrium allocation is not necessarily fair in the sense of "envy-freeness." For instance, if one agent is endowed with more of each good than another, then he has a larger budget set at every price vector and the agent with the smaller endowment may envy him in equilibrium. However, a competitive equilibrium allocation does satisfy the following criterion of fairness: if two agents have the same endowment, then neither envies the other. In the model that I have presented, an agent's time is his endowment. If two agents are of the same kind so that everyone is indifferent between them, then these two agents have the same endowment of time. Since prices are indexed by kinds rather than identities of agents, at a DIP equilibrium allocation, no agent envies another agent of the same kind who has the same endowment of money.

[19] Recall that Bondareva and Shapley's "balancedness" condition (Bondareva, 1963; Shapley, 1967) only guarantees nonemptiness of the weak core (Scarf, 1967).

[20] An example is available upon request.

[21] Though, it is necessarily *weakly* Pareto-efficient: there is no way to make *every* agent better off.

[22] As long as each agent has a unique maximizer in his budget set (for instance, if utility functions are *strictly* concave), every εDIP equilibrium is cost minimizing.

[23] The idea of partitioning the agents in this way has appeared before. Conley and Wooders (2001) allow this to be a choice and call it a "crowding type" while Ellickson et al. (1999, 2001) take it to be exogenous, as I have suggested here, and call it an "external characteristic."

Dropping the bipartite structure When the set of agents can be partitioned into two sets who are to be matched, the Birkhoff–von Neumann theorem (Birkhoff, 1946; von Neumann, 1953) says that every feasible fractional allocation can be interpreted as a lottery over one-to-one matches. This makes the application to problems of randomized matching and scheduling straightforward: the fractional assignment can actually be manifested as a schedule or grand lottery over matches. The assumption of a bi-partition is, however, more than what is needed. Budish et al. (2013) provide a more general condition.

The analysis presented in this paper, however, does not rely on this assumption. It is merely used for clarity and to enable the interpretation and applicability of the model. For applications that are different from randomized matching or contiguous scheduling, like the market for swapped assets,[24] this assumption is not needed.

Larger coalitions than pairs Most of the coalition formation literature deals with "binary" membership: a person is either part of a coalition or he is not. To my knowledge there is no previous work on fractional coalitions outside of Garratt and Qin (1996) and Biró and Fleiner (2012). The former study an extension of "market games" (Shapley and Shubik, 1969) to lotteries over coalitions. That is a quasilinear setting where feasible allocations are only lotteries over partitions of the set of agents. The latter use a variant of Scarf's Lemma to show that the "fractional core" is not empty.

Recently Hatfield and Kominers (2015) have studied a related model of "multilateral matching" where agents form groups to undertake joint ventures. However, they assume quasilinear preferences with concave valuation functions and show the existence of competitive allocations and prove the related welfare theorems.

The results of the current paper generalize to larger coalitions than pairs in a very straightforward manner. The main difference is in the proof of Theorem 1: the definition of excess demand is slightly different. Thus, Step 2 of the proof requires some extra work. The rest of the analysis follows more or less unchanged.

A further extension, which is easily accommodated, is to index partnership by "contracts." That is, agents i and j being matched under contract t is different from the same pair being matched under contract t'.

Appendix A. Proof of Theorem 1

Since the argument is based on Kakutani's fixed point theorem, I restrict attention to the following compact set of prices.

$$\bar{\mathbb{P}} \equiv \{p \in \mathbb{P} : ||p|| \leq 1\}.$$[25]

Let $(u, \omega) \in \mathcal{E}$ be such that for each $i \in N$, $\omega_i > 0$.

I start by defining, for each $i \in N$, the correspondence $F_i : \bar{\mathbb{P}} \rightrightarrows X_i$ by setting for each $p \in \bar{\mathbb{P}}$,

$$F_i(p) \equiv \left\{ (\pi_i, x_i) \in X_i : \begin{array}{l} \pi_i \cdot p_i + x_i p_0 \leq \omega_i p_0 + \frac{1-||p||}{n} \\ \text{and for each } (\pi_i', x_i') \text{ such that} \\ \pi_i' \cdot p_i + x_i' p_0 < \omega_i p_0 + \frac{1-||p||}{n} \\ u_i(\pi_i, x_i) \geq u_i(\pi_i', x_i') \end{array} \right\}.$$

Think of F_i in the following way: Each agent's budget constraint is relaxed by $\frac{1-||p||}{n}$. The set $F_i(p)$ consists of the bundles in this relaxed budget set that are at least as good for i as any bundle in its *interior*.[26]

Before I define a correspondence, $F_0 : X \rightrightarrows \bar{\mathbb{P}}$, for the "market maker," I define the "excess demand" function, $d(\pi, x)$. For each $(\pi, x) \in X$ the excess demand for money is obvious: $d_0(\pi, x) \equiv \sum_{i\in N} x_i - \sum_{i \in N} \omega_i$. For each pair $i, j \in N$, such that $i \in M$ if and only if $j \in W$, the excess demand of i for j is $d_{ij}(\pi, x) \equiv \pi_{ij} - \pi_{ji}$. Since, by definition, $d_{ij}(\pi, x) = -d_{ji}(\pi, x)$ and $d_{ii}(\pi, x) = 0$, we can identify $d(\pi, x)$ with an element of $\mathbb{R}^{M\times W \cup \{0\}}$. Thus, if $d(\pi, x) \neq 0$ then $\frac{d(\pi,x)}{||d(\pi,x)||} \in \bar{\mathbb{P}}$. Define $F_0(\pi, x)$ as follows:

$$F_0(\pi, x) \equiv \begin{cases} \left\{ \frac{d(\pi,x)}{||d(\pi,x)||} \right\} & \text{if } d(\pi, x) \neq 0 \text{ or} \\ \bar{\mathbb{P}} & \text{otherwise.} \end{cases}$$

Define $F : X \times \bar{\mathbb{P}} \rightrightarrows X \times \bar{\mathbb{P}}$ by setting for each $((\pi, x), p) \in X \times \bar{\mathbb{P}}$,

$$F((\pi, x), p) \equiv \bigtimes_{i\in N} F_i(p) \times F_0(\pi, x).$$

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[24] See, for instance, Atkeson et al. (2013).

[25] I denote the Euclidean norm of $p \in \mathbb{P}$ by $||p||$.

[26] This trick of relaxing the budget set by $\frac{1-||p||}{n}$ was developed by Bergstrom (1976) to handle the lack of free disposability. See Shafer (1976) and Mas-Colell (1992) for other examples where it is used.

To apply Kakutani's fixed point theorem, F_0 and, for each $i \in N$, F_i need to be upper hemicontinuous and nonempty, closed, and convex valued. These properties are immediate from the definition of F_0. I check them for F_i.

Upper hemicontinuity: Let, for each $\nu \in \mathbb{N}$, $p^\nu \in \bar{\mathbb{P}}$ be such that $p^\nu \to p \in \bar{\mathbb{P}}$. Let $(\pi_i^\nu, x_i^\nu) \in F_i(p_i^\nu)$ be such that $(\pi_i^\nu, x_i^\nu) \to (\pi_i, x_i) \in X_i$. If $(\pi_i, x_i) \notin F_i(p)$, then there are two cases.

Case 1. $(\pi_i, x_i) \cdot p_i > \omega_i p_0 + \frac{1-||p||}{n}$. Then, there is $\underline{\nu}$ such that for each $\nu > \underline{\nu}$, $(\pi_i^\nu, x_i^\nu) \cdot p_i > \omega_i p_0 + \frac{1-||p||}{n}$. In turn, there is $\overline{\nu} \geq \underline{\nu}$ such that for each $\nu > \overline{\nu}$, $(\pi_i^\nu, x_i^\nu) \cdot p_i^\nu > \omega_i p_0^\nu + \frac{1-||p^\nu||}{n}$. This contradicts, for $\nu > \overline{\nu}$, the assumption that $(\pi_i^\nu, x_i^\nu) \in F_i(p^\nu)$.

Case 2: There is $(\pi_i', x_i') \in X_i$ such that $u_i(\pi_i', x_i') > u_i(\pi_i, x_i)$ and $(\pi_i', x_i') \cdot p_i < \omega_i p_0 + \frac{1-||p||}{n}$. By continuity of u_i, there is $\underline{\nu}$ such that for each $\nu > \underline{\nu}$, $u_i(\pi_i', x_i') > u_i(\pi_i^\nu, x_i^\nu)$. There is also $\overline{\nu}$ such that for each $\nu > \overline{\nu}$, $(\pi_i', x_i') \cdot p_i^\nu < \omega_i p_0^\nu + \frac{1-||p^\nu||}{n}$. For $\nu > \max\{\underline{\nu}, \overline{\nu}\}$, this contradicts $(\pi_i^\nu, x_i^\nu) \in F_i(p^\nu)$.

Thus, F_i is upper hemicontinuous.

Nonempty-valuedness: The set $\{(\pi_i, x_i) \in X_i : (\pi_i, x_i) \cdot p_i \leq \omega_i p_0 + \frac{1-||p||}{n}\}$ is compact. Since u_i is continuous, it has a maximal element (π_i^*, x_i^*) in this set. Since (π_i^*, x_i^*) is maximal in the relaxed budget set, $(\pi_i^*, x_i^*) \in F_i(p)$.

Closed-valuedness: This is obvious from continuity of u_i.

Convex-valuedness: Let the pair $(\underline{\pi}_i, \underline{x}_i), (\overline{\pi}_i, \overline{x}_i) \in F_i(p)$. Let (π_i, x_i) be a convex combination of $(\underline{\pi}_i, \underline{x}_i)$ and $(\overline{\pi}_i, \overline{x}_i)$. By quasi-concavity of u_i, $u_i(\pi_i, x_i) \geq u_i(\underline{\pi}_i, \underline{x}_i)$ or $u_i(\pi_i, x_i) \geq u_i(\overline{\pi}_i, \overline{x}_i)$. Without loss of generality, suppose the latter is true. Since $(\underline{\pi}_i, \underline{x}_i) \cdot p_i \leq \omega_i p_0 + \frac{1-||p||}{n}$ and $(\overline{\pi}_i, \overline{x}_i) \cdot p_i \leq \omega_i p_0 + \frac{1-||p||}{n}$ it follows that $(\pi_i, x_i) \cdot p_i \leq \omega_i p_0 + \frac{1-||p||}{n}$. If there is (π_i', x_i') such that $(\pi_i', x_i') \cdot p_i < \omega_i p_0 + \frac{1-||p||}{n}$ and $u_i(\pi_i', x_i') > u_i(\pi_i, x_i)$ then $u_i(\pi_i', x_i') > u_i(\overline{\pi}_i, \overline{x}_i)$. This contradicts the assumption that $(\overline{\pi}_i, \overline{x}_i) \in F_i(p)$. Thus, $(\pi_i, x_i) \in F_i(p)$.

By Kakutani's fixed point theorem, F has a fixed point $((\pi, x), p)$. I contend that this is a DIP equilibrium of (u, ω). I establish this in four steps.

Step 1. Show that $||p|| = 1$: For each $i \in N$, it is immediate from monotonicity of u_i in money, that

$$(\pi_i, x_i) \cdot p_i = \omega_i p_0 + \frac{1 - ||p||}{n}. \tag{1}$$

If $||p|| \neq 1$, by definition of F_0, $d(\pi, x) = 0$.

Summing (1) over all $i \in N$,

$$\sum_{i \in N} (\pi_i, x_i) \cdot p_i = \sum_{i \in N} \omega_i p_0 + 1 - ||p||.$$

Or

$$\sum_{m \in M} \sum_{w \in W} \pi_{mw} p_{mw} + \sum_{w \in W} \sum_{m \in M} \pi_{wm} p_{wm} + \left(\sum_{i \in N} x_i - \sum_{i \in N} \omega_i \right) p_0 = 1 - ||p||.$$

Since $d(\pi, x) = 0$, $\sum_{i \in N} x_i - \sum_{i \in N} \omega_i = 0$. Thus,

$$\sum_{m \in M} \sum_{w \in W} \pi_{mw} p_{mw} + \pi_{wm} p_{wm} = 1 - ||p||.$$

That is,

$$\sum_{m \in M} \sum_{w \in W} p_{mw} (\pi_{mw} - \pi_{wm}) = 1 - ||p||.$$

Since for each $m \in M$ and $w \in W$, $d_{mw}(\pi, x) = 0$, m demands the same amount of w's time as w demands of m's time. That is, $\pi_{mw} = \pi_{wm}$. Thus, $1 - ||p|| = 0$. This contradicts the assumption that $||p|| < 1$.

Step 2. Show that $d(\pi, x) = 0$: If not, by definition of F_0, $p = \frac{d(\pi, x)}{||d(\pi, x)||}$. Then,

$$d(\pi, x) \cdot p = d(\pi, x) \cdot \frac{d(\pi, x)}{||d(\pi, x)||} = ||d(\pi, x)|| > 0.$$

By Step 1, $||p|| = 1$. Thus, for each $i \in N$,

$$(\pi_i, x_i) \cdot p_i \leq \omega_i p_0. \tag{2}$$

Summing (2) over all $i \in N$,

$$\sum_{i\in N}(\pi_i, x_i)\cdot p_i \le \sum_{i\in N}\omega_i p_0.$$

That is,

$$\sum_{m\in M}\sum_{w\in W}(\pi_{mw}p_{wm}+\pi_{wm}p_{wm})+\sum_{i\in N}(x_i-\omega_i)p_0 \le 0.$$

So

$$\sum_{m\in M}\sum_{w\in W} d_{mw}(\pi, x)p_{mw} + d_0(\pi, x)p_0 \le 0.$$

In other words, $d(\pi, x)\cdot p \le 0$, which contradicts the earlier assertion that $d(\pi, x)\cdot p > 0$. Thus, $d(\pi, x) = 0$.

Step 3. Show that $p_0 > 0$: It is easy to see that $p_0 \ge 0$ for if $p_0 < 0$, for any $\alpha \in \mathbb{R}_+$ such that $\alpha > 0$, $(\pi_i, x_i+\alpha)\cdot p_i < (\pi_i, x_i)\cdot p_i \le \omega_i p_0 + \frac{1-||p||}{n}$. Since u_i is monotonic in money, $u_i(\pi_i, x_i+\alpha) > u_i(\pi_i, x_i)$. This contradicts $(\pi_i, x_i) \in F_i(p)$.

By Step 1, $||p|| = 1$. If $p_0 = 0$ then there is at least one pair $m \in M$ and $w \in W$ such that $p_{mw} \neq 0$. Then either $p_{mw} < 0$ or $p_{wm} < 0$. Without loss of generality, suppose that $p_{mw} < 0$. Define $\hat{\pi}_m \in \Delta_m$ so that $\hat{\pi}_{mw} = 1$. Then, $\hat{\pi}_{mw}p_{mw} < 0$. Let, for each $\gamma \in (0, 1)$, $\pi_m^\gamma = \gamma\pi_m + (1-\gamma)\hat{\pi}_m$. Then, for every $\alpha \in \mathbb{R}_+$, $(\pi_m^\gamma, \alpha)\cdot p_m < 0$. Since u_m is continuous and monotonic in money, there are $\gamma \in (0, 1)$ and $\alpha \in \mathbb{R}_+$ such that and $u_m(\pi_m^\gamma, \alpha) > u_m(\pi_m, x_m)$. This contradicts $(\pi_m, x_m) \in F_m(p)$. Thus, $p_0 > 0$.

Step 4. Show that for each $i \in N$, (π_i, x_i) is maximal in $B_i(p_i, \omega_i)$: Let $\delta_i \in \Delta_i$ be such that $\delta_{ii} = 1$. By Step 3, $p_0 > 0$. Thus, $\delta_i \cdot p_{ii} = 0 < \omega_i p_0$. Suppose there is $(\pi_i', x_i') \in B_i(p, \omega_i)$ such that $u_i(\pi_i', x_i') > u_i(\pi_i, x_i)$. Since $(\pi_i', x_i') \in B_i(p, \omega_i)$ and $(\pi_i, x_i) \in F_i(p)$, by definition of F_i, $(\pi_i', x_i')\cdot p_i = \omega_i p_0$. For each $\gamma \in (0, 1)$, let

$$(\pi_i^\gamma, x_i^\gamma) \equiv (\gamma\pi_i' + (1-\gamma)\delta_i, \gamma x_i').$$

Then, for each $\gamma \in (0, 1)$, $(\pi_i^\gamma, x_i^\gamma)\cdot p_i < \omega_i p_0$. Since u_i is continuous, for γ close to 1, $u_i(\pi_i^\gamma, x_i^\gamma) > u_i(\pi_i, x_i)$. This contradicts $(\pi_i, x_i) \in F_i(p)$.

Thus, since $d(\pi, x) = 0$ and for each $i \in N$, (π_i, x_i) maximizes u_i in $B_i(p)$, $((\pi, x), p)$ is a DIP equilibrium. □

Appendix B. Proof of Theorem 2

Let $u \in \hat{\mathcal{U}}$. For each $i \in N$, define $\tilde{u}_i : X_i \to \mathbb{R}$ by setting for each $(\pi_i, x_i) \in X_i$, $\tilde{u}_i(\pi_i, x_i) \equiv u_i(\pi_i) + x_i$. Let $\omega \in \mathbb{R}^N$ be such that for each $i \in N$, $\omega_i > 0$ and $\sum_{i\in N}\omega_i = \varepsilon$. It is clear that $\tilde{u}_i$ is continuous and concave (since u_i is). It is also monotonic in money. Since the conditions of Theorem 1 are met, the economy $(\tilde{u}, \omega)$ has a DIP equilibrium $((\pi, x), \tilde{p}) \in Z(\omega)\times\mathbb{P}$. Since $(\pi, x) \in Z(\omega)$, $\pi \in \Pi$. Let $p \in \hat{\mathbb{P}}$ be such that $p = \tilde{p}_{-0}$ (this is $\tilde{p}$ without $\tilde{p}_0$). For each $i \in N$, let $t_i \equiv (\omega_i - x_i)\tilde{p}_0$.

I first show that (π, p, t) is an εDIP equilibrium of u. I then verify that it is cost minimizing.

Since $\sum_{i\in N} x_i = \sum_{i\in N}\omega_i$, $\sum_{i\in N} t_i = \sum_{i\in N}(\omega_i - x_i)\tilde{p}_0 = 0$.

To show that (π, p, t) is an εDIP equilibrium, there are three conditions to check for each $i \in N$.

a) $\pi_i \in \underset{\hat{B}_i(p,t_i)}{\operatorname{argmax}}\, u_i$: Suppose there is $\pi_i' \in \hat{B}_i(p, t_i)$ such that $u_i(\pi_i') > u_i(\pi_i)$. Then, $\pi_i'\cdot p_i \le t_i$. So $\pi_i'\cdot p_i \le (\omega_i - x_i)\tilde{p}_0$. Thus, $(\pi_i', x_i)\cdot\tilde{p}_i \le \omega_i\tilde{p}_0$. In other words, $(\pi_i', x_i) \in B_i(\tilde{p}, \omega_i)$ and

$$\tilde{u}_i(\pi_i', x_i) = u_i(\pi_i') + x_i > u_i(\pi_i) + x_i = \tilde{u}_i(\pi_i, x_i).$$

This contradicts the assumption that $((\pi, x), \tilde{p})$ is a DIP equilibrium of $(\tilde{u}, \omega)$.

b) $\pi_i\cdot p_i = t_i$: Since $\tilde{u}_i$ is monotonic in money, $\pi_i\cdot p_i + x_i\tilde{p}_0 = \omega_i\tilde{p}_0$. So $\pi_i\cdot p_i = (\omega_i - x_i)\tilde{p}_0 = t_i$.

c) For each $\pi_i' \in \Delta_i$, if $\pi_i'\cdot p_i \le 0$ then $u_i(\pi_i) \ge u_i(\pi_i') - \varepsilon$: Since $(\pi, x) \in Z(\omega)$, $x_i \le \sum_{i\in N}\omega_i = \varepsilon$. For each $\pi' \in \Delta_i$ such that $\pi_i'\cdot p_i \le 0$ we have that $\pi_i'\cdot p_i \le \omega_i\tilde{p}_0$ so that $(\pi_i', 0) \in B_i(\tilde{p}, \omega_i)$. If $u(\pi_i') > u(\pi_i) + \varepsilon$ then

$$\tilde{u}_i(\pi_i', 0) = u_i(\pi_i') + 0 > u_i(\pi_i) + \varepsilon \ge u_i(\pi_i) + x_i = \tilde{u}_i(\pi_i, x_i).$$

This contradicts the assumption that $((\pi, x), \tilde{p})$ is a DIP equilibrium for $(\tilde{u}, \omega)$.

Finally, I show that (π, p, t) is cost minimizing: Since, for each $i \in N$, $\tilde{u}_i$ is monotonic in money, $\tilde{p}_0 > 0$. Suppose $\pi_i' \in \underset{\hat{B}_i(p,t_i)}{\operatorname{argmax}}\, u_i$, and that $\pi_i'\cdot p_i < \pi_i\cdot p_i$. Let $x_i' = x_i + \frac{\pi_i\cdot p_i - \pi_i'\cdot p_i}{\tilde{p}_0}$. Then, $\tilde{u}_i(\pi_i', x_i') = u_i(\pi_i') + x_i' > u_i(\pi_i) + x_i$ and $(\pi_i', x_i') \in B_i(\tilde{p}, \omega)$. This contradicts the assumption that $((\pi, x), p)$ is a DIP equilibrium of $(\tilde{u}, \omega)$. □

Appendix C. Proofs of Propositions 1, 2, and 3

C.1. Proof of Proposition 1

Let $(u, \omega) \in \mathcal{E}$ and let $((\pi, x), p) \in Z(\omega) \times \mathbb{P}$ be a DIP equilibrium at (u, ω). Suppose that $N' \subseteq N$ blocks (π, x) via $(\pi', x') \in Z(\omega)$. For each $i \in N'$, since u_i is monotonic in money, $(\pi_i, x_i) \cdot p_i = \omega_i p_0$. Since $u_i(\pi_i', x_i') \geq u_i(\pi_i, x_i)$, I deduce that

$$(\pi_i', x_i') \cdot p_i \geq \omega_i p_0. \tag{3}$$

Inequality (3) is strict for at least one $i \in N'$. Summing (3) over all $i \in N'$,

$$\sum_{i \in N'} (\pi_i', x_i') \cdot p_i > \sum_{i \in N'} \omega_i p_0.$$

Since $\sum_{i \in N'} x_i' = \sum_{i \in N'} \omega_i$,

$$\sum_{m \in M \cap N'} \sum_{w \in W \cap N'} \left(\pi'_{mw} p_{mw} + \pi'_{wm} p_{wm}\right) > 0.$$

That is,

$$\sum_{m \in M \cap N'} \sum_{w \in W \cap N'} p_{mw} \left(\pi'_{mw} - \pi'_{wm}\right) > 0.$$

This contradicts $\pi' \in \Pi$ which requires that for each pair $m \in M$ and $w \in W$, $\pi'_{mw} = \pi'_{wm}$. Thus (π, x) is in the core. □

C.2. Proof of Proposition 2

Let $u \in \hat{\mathcal{U}}$ and let $(\pi, p, t) \in \Pi \times \hat{\mathbb{P}} \times \mathbb{R}^N$ be a cost minimizing εDIP equilibrium at u. Suppose that $\pi' \in \Pi$ Pareto-dominates π at u. For each $i \in N$, if $u_i(\pi_i') > u_i(\pi_i)$ then $\pi_i' \cdot p_i > t_i$. If $u_i(\pi_i') = u_i(\pi_i)$, then since (π, p) is cost minimizing, $\pi_i' \cdot p_i \geq \pi_i \cdot p_i = t_i$. Summing over $i \in N$,

$$\sum_{i \in N} \pi_i' \cdot p_i > \sum_{i \in N} t_i = 0.$$

Rearranging this,

$$\sum_{m \in M} \sum_{w \in W} p_{mw} (\pi'_{mw} - \pi'_{wm}) > \sum_{i \in N} t_i = 0.$$

However, this contradicts $\pi' \in \Pi$ which requires that for each pair $m \in M$ and $w \in W$ $\pi'_{mw} = \pi'_{wm}$. Thus π is Pareto-efficient. □

C.3. Proof of Proposition 3

Let $u \in \hat{\mathcal{U}}$ and let $(\pi, p, t) \in \Pi \times \mathbb{P} \times \mathbb{R}^N$ be an εDIP equilibrium at u. Suppose that $N' \subseteq N$ ε-strongly blocks π at u via $\pi' \in \Pi$. Then, for each $i \in N'$, $\pi_i' \cdot p_i > 0$. Summing this over all $i \in N'$,

$$\sum_{i \in N'} \pi_i' p_i > 0. \tag{4}$$

Rearranging (4), yields a contradiction just as in the proof of Proposition 1. □

Appendix D. Weak monotonicity of preferences in money

A very natural question is to ask whether the assumption of strict monotonicity is necessary to show the existence of DIP equilibria. After all, if DIP equilibria exist when money is a neutral good, I could follow the approach of Hylland and Zeckhauser (1979) and endow each agent with fake money and consider *exact* rather than approximate DIP equilibria for economies without money.

In this section, I present an example of an economy where preferences are such that no agent values money but the remaining assumptions of continuity and quasi-concavity are satisfied. I show that this economy has no DIP equilibrium.

Example 2 *(No DIP equilibrium if money is a neutral good).* Let $M \equiv \{m_1, m_2\}$ and $W \equiv \{w_1, w_2\}$. For each $i \in N$, let $\omega_i > 0$. Let u be a profile of continuous and quasi-concave utility functions that are weakly monotonic in money, be such that m_1 only values his time alone, m_2 only values his time with w_2, w_1 values her time with m_1 and dislikes spending time with m_2, and w_2 only values her how close to $\frac{1}{2}$ her time spent with m_2 is. That is,

$$u_{m_1}(\pi_{m_1}, x_{m_1}) = \pi_{m_1 m_1},$$

$$u_{m_2}(\pi_{m_2}, x_{m_2}) = \pi_{m_2 w_2},$$

$$u_{w_1}(\pi_{w_1}, x_{w_1}) = \pi_{w_1 m_1} - \pi_{w_1 m_2}, \quad \text{and}$$

$$u_{w_2}(\pi_{w_2}, x_{2_2}) = -(\pi_{w_2 m_2} - \tfrac{1}{2})^2.$$

Given $p \in \mathbb{P}$,

$$B_{m_1}(p) = \{(\pi_{m_1}, x_{m_1}) \in X_{m_1} : \pi_{m_1 w_1} p_{m_1 w_1} + \pi_{m_1 w_2} p_{m_1 w_2} + x_{m_1} p_0 \leq \omega_{m_1} p_0\},$$

$$B_{m_2}(p) = \{(\pi_{m_2}, x_{m_2}) \in X_{m_2} : \pi_{m_2 w_1} p_{m_2 w_1} + \pi_{m_2 w_2} p_{m_2 w_2} + x_{m_2} p_0 \leq \omega_{m_2} p_0\},$$

$$B_{w_1}(p) = \{(\pi_{w_1}, x_{w_1}) \in X_{w_1} : \pi_{w_1 m_1} p_{w_1 m_1} + \pi_{w_1 m_2} p_{w_1 m_2} + x_{w_1} p_0 \leq \omega_{w_1} p_0\}, \quad \text{and}$$

$$B_{w_2}(p) = \{(\pi_{w_2}, x_{w_2}) \in X_{w_2} : \pi_{w_2 m_1} p_{w_2 m_1} + \pi_{w_2 m_2} p_{w_2 m_2} + x_{w_2} p_0 \leq \omega_{w_2} p_0\}.$$

I show that this economy does not have a DIP equilibrium. For the sake of contradiction, suppose that $((\pi^*, x^*), p^*)$ is a DIP equilibrium.

First, consider m_1's problem. Let $\pi_{m_1} \in \Delta_{m_1}$ be such that $\pi_{m_1 m_1} = 1$. Regardless of p^*, $(\pi_{m_1}, \omega_{m_1}) \in B_{m_1}(p^*)$. Furthermore, for each $\pi'_{m_1} \in \Delta_{m_1} \setminus \{\pi_{m_1}\}$ and each $x'_{m_1} \in \mathbb{R}_+$, $u_{m_1}(\pi_{m_1}, \omega_{m_1}) > u_{m_1}(\pi'_{m_1}, x'_{m_1})$. Thus, $\pi^*_{m_1} = \pi_{m_1}$.

Since $\pi^* \in \Pi$, we have that $\pi^*_{w_1 m_1} = \pi^*_{w_2 m_1} = 0$.

Next, I consider w_1's problem. Since $\pi^*_{w_1 m_1} = 0$, I deduce that $\pi^*_{w_1 w_1} = 1$. Otherwise, if $\pi^*_{w_1 w_1} < 1$, then $\pi^*_{w_1 m_2} > 0$ so that $u_{w_1}(\pi^*_{w_1}, x^*_{w_1}) < 0$. Letting $\pi_{w_1} \in \Delta_{w_1}$ be such that $\pi_{w_1 w_1} = 1$, $(\pi_{w_1}, \omega_{w_1}) \in B_{w_1}(p^*)$ and, by definition of u_{w_1}, $u_{w_1}(\pi_{w_1}, \omega_{w_1}) = 0 > u_{w_1}(\pi^*_{w_1}, x^*_{w_1})$. This contradicts the assumption that $((\pi^*, x^*), p^*)$ is a DIP equilibrium.

I now show that $p^*_0 \leq 0$. If $p^*_{w_1 m_1} \leq 0$, by definition of u_{w_1}, $\pi^*_{w_1 m_1} = 1$, which is a contradiction. Thus, $p^*_{w_1 m_1} > 0$. If $p^*_0 > 0$, consider $(\pi_{w_1}, x_{w_1}) \in X_{w_1}$ such that $\pi_{w_1 m_1} = \min\{1, \frac{\omega_{w_1} p^*_0}{p^*_{w_1 m_1}}\}$ and $\pi_{w_1 w_1} = 1 - \pi_{w_1 m_1}$. Since p^*_0 and $p^*_{w_1 m_1} > 0$, $\pi_{w_1 m_1} > 0$. Let $x_{w_1} = 0$. By definition of u_{w_1}, since $\pi^*_{w_1 m_1} = 0$, $u_{w_1}(\pi_{w_1}, x_{w_1}) > u_{w_1}(\pi^*_{w_1}, x^*_{w_1})$ regardless of $x^*_{w_1}$. However, $(\pi_{w_1}, x_{w_1}) \in B_{w_1}(p^*)$. This contradicts the assumption that $((\pi^*, x^*), p^*)$ is a DIP equilibrium. Thus, $p^*_0 \leq 0$.

To solve m_2 and w_2's problems, I consider three cases based on $p^*_{m_2 w_2}$. In each case, I contradict $\pi^* \in \Pi$. The analysis is essentially identical to that of Example 1, the difference being that I take care to account for the additional prices.

a) Suppose $p^*_{m_2 w_2} = 0$. Let $\pi_{m_2} \in \Delta_{m_2}$ be such that $\pi_{m_2 w_2} = 1$. Since $p^*_{m_2 w_2} = 0$, $(\pi_{m_2}, \omega_{m_2}) \in B_{m_2}(p^*)$. For each $\pi'_{m_2} \in \Delta_{m_2} \setminus \{\pi_{m_2}\}$ and each $x'_{m_2} \in \mathbb{R}_+$, by definition of u_{m_2}, $u_{m_2}(\pi_{m_2}, \omega_{m_2}) > u_{m_2}(\pi'_{m_2}, x'_{m_2})$. Thus, $\pi^*_{m_2 w_2} = \pi_{m_2 w_2} = 1$.
   Let $\pi_{w_2} \in \Delta_{w_2}$ be such that $\pi_{w_2 m_2} = \frac{1}{2}$ and $\pi_{w_2 w_2} = \frac{1}{2}$. Since $p^*_{w_2 m_2} = 0$, $(\pi_{w_2}, \omega_{w_2}) \in B_{w_2}(p^*)$. For each $\pi'_{w_2} \in \Delta_{w_2} \setminus \{\pi_{w_2}\}$ and each $x'_{w_2} \in \mathbb{R}_+$, by definition of u_{w_2}, $u_{w_2}(\pi_{w_2}, \omega_{w_2}) > u_{w_2}(\pi'_{w_2}, x'_{w_2})$. Thus, $\pi^*_{w_2 m_2} = \pi_{w_2 m_2} = \frac{1}{2}$.
   So $\pi^*_{m_2 w_2} = 1$ but $\pi^*_{w_2 m_2} = \frac{1}{2}$, contradicting $\pi^* \in \Pi$.
b) Suppose $p^*_{m_2 w_2} < 0$. By an identical argument as the case of $p^*_{m_2 w_2} = 0$, we have that $\pi^*_{m_2 w_2} = 1$.
   Since $p^*_0 \leq 0$ and $p^*_{w_2 m_2} = -p^*_{m_2 w_2} > 0$, regardless of $p^*_{w_2 m_1}$, there is no $(\pi_{w_2}, x_{w_2}) \in B_{w_2}(p^*)$ such that $\pi_{w_2 m_1} = 0$ and $\pi_{w_2 m_2} > 0$. Thus, since $\pi^*_{w_2 m_1} = 0$, $\pi^*_{w_2 m_2} = 0$.
   So $\pi^*_{m_2 w_2} = 1$ but $\pi^*_{w_2 m_2} = 0$, contradicting $\pi \in \Pi$.
c) Suppose $p^*_{m_2 w_2} > 0$. Then $p^*_{w_2 m_2} = -p^*_{m_2 w_2} < 0$. By an identical argument as the case of $p^*_{m_2 w_2} = 0$, we have that $\pi^*_{w_2 m_2} = \frac{1}{2}$.
   Since $p^*_0 \leq 0$ and $p^*_{m_2 w_2} > 0$, regardless of $p^*_{m_2 w_1}$, there is no $(\pi_{m_2}, x_{m_2}) \in B_{m_2}(p^*)$ such that $\pi_{m_2 w_1} = 0$ and $\pi_{m_2 w_2} > 0$. Thus, since $\pi^*_{w_1 m_2} = 0 = \pi^*_{m_2 w_1}$, $\pi^*_{m_2 w_2} = 0$.
   So $\pi^*_{w_2 m_2} = \frac{1}{2}$ but $\pi^*_{m_2 w_2} = 0$, contradicting $\pi \in \Pi$.

The problem with this economy is that we can think of it as the union of two sub-economies: the first consists of m_1 and w_1 while the second consists of m_2 and w_2. Neither group has anything to offer that the other values. Effectively, the only price that connects them is the price of fake money. If that price is positive, so that each agent is guaranteed a positive amount of wealth, then there is no way to prevent w_1 from buying some of m_1's time. However, m_1 is not interested in spending any of his time with her since, regardless of the prices, his bliss point of staying alone is affordable to him. So an equilibrium for this sub-economy requires the fake money to have a non-positive price. On the other hand, the second sub-economy (which is the one in Example 1) has no equilibrium without each agent being endowed with positive wealth.

Appendix E. Generalizations from Section 6

In this section, I describe a more general model than the one in Section 2, extend the definition of DIP equilibrium, and prove a counterpart of Theorem 1 for it. The proofs of counterparts to the remaining results in the paper are identical to the ones in the body of the paper, so I have omitted them.

E.1. The model

Let N be the set of agents and K be the set of *kinds*.[27] For each $i \in N$, his availability is $a_i \in \mathbb{R}_+$ and his kind is $\kappa_i \in K$. Let $\{N_\kappa\}_{\kappa \in K}$ be a partition of N such that for each $\kappa \in K$, N_κ consists of all of the agents who are of kind κ. Let $\mathcal{S} \subseteq 2^K$ be the set all feasible *partnership-kinds*. I require that $\mathcal{S}$ contains all singletons. Each $T \in \mathcal{S}$ corresponds to the sort of partnership that consists of one agent of each kind in T. For each $\kappa \in K$, let $\mathcal{S}_\kappa \subseteq \mathcal{S}$ be the partnership-kinds involving kind κ. Thus, each $i \in N_\kappa$ may be in a partnership $G \subseteq N$ such that $i \in G$ if there is $T \in \mathcal{S}_\kappa$ such that G contains one member of each kind in T.

Since only the kind of a partnership and not the identities of its members matters, i's partnership consumption set is $\Delta_i \equiv \{x_i \in \mathbb{R}_+^{\mathcal{S}_{\kappa_i}} : \sum_{T \in \mathcal{S}_{\kappa_i}} x_{iT} = a_i\}$. Let $\Delta \equiv \times_{i \in N} \Delta_i$. A feasible partnership profile is $\pi \in \Delta$ such that for each $T \in \mathcal{S}$ and each pair $\kappa, \theta \in T$,

$$\sum_{i \in N_\kappa} \pi_{iT} = \sum_{i \in N_\theta} \pi_{iT}.$$[28]

That is, the total amount of partnership-kind T consumed by agents of each member kind is the same. Let Π be the set of feasible partnership profiles.

Leaving fixed, for each $i \in N$, $\kappa_i \in K$ and $a_i \in \mathbb{R}_+$ and for each $\kappa \in K$, $\mathcal{S}_\kappa \subset 2^K$, an economy is fully described by $u \in \mathcal{U}$ and $\omega \in \mathbb{R}_+^N$. Let $\mathcal{E} \equiv \mathcal{U} \times \mathbb{R}_+^N$ be the set of economies.

E.2. DIP equilibria

For each $\kappa \in K$ and each $T \in \mathcal{S}_\kappa$, the price $p_{\kappa T} \in \mathbb{R}$ is what an agent of kind κ pays for each unit of partnership-kind T that he consumes. A positive price means that agents of a particular kind need to pay in order to be in a certain kind of partnership. A negative price signifies that they are paid to be in it. Define the set of price indices as

$$I \equiv \{(\kappa, T) \in K \times \mathcal{S} : T \in \mathcal{S}_\kappa\}.$$

I require price vectors to be such that members of a partnership only pay one another. That is, I restrict price vectors to the following set:

$$\mathbb{P} \equiv \left\{ p \in \mathbb{R}^{I \cup \{0\}} : \text{ for each } T \in \mathcal{S}, \sum_{\kappa \in T} p_{\kappa T} = 0 \right\}.$$

This is an intuitive restriction on prices. After all, if an agent of one kind pays to be in a partnership, what he pays ought to be received by the others. One can interpret these prices as being the difference between a positive amount paid by each member of a partnership and a rebate of what is collected. It is this restriction on prices that guarantees the welfare properties of equilibrium allocations.[29]

For each $p \in \mathbb{P}$ and each $i \in N$, i's budget set is

$$B_i(p, \omega_i) \equiv \left\{ (\pi_i, x_i) \in X_i : \sum_{T \in \mathcal{S}_{\kappa_i}} \pi_{iT} p_{\kappa_i T} + x_i p_0 \le \omega_i p_0 \right\}.$$

Henceforth, I denote $\sum_{T \in \mathcal{S}_{\kappa_i}} \pi_{iT} p_{\kappa_i T}$ by $\pi_i \cdot p_{\kappa_i}$.

For each $(u, \omega) \in \mathcal{E}$, $((\pi, x), p) \in X \times \mathbb{P}$ is a DIP equilibrium if for each $i \in N$, $(\pi_i, x_i) \in \underset{B_i(p,\omega_i)}{\operatorname{argmax}}\ u_i$ and $(\pi, x) \in Z(\omega)$.

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[27] I have chosen the term "kind" as opposed to "type" since an agent's kind does not dictate his preferences. Common usage of the word type is that it encapsulates all relevant information about an agent, including his preferences. Conley and Wooders (2001) allow this to be a choice and call it a "crowding type" while Ellickson et al. (1999, 2001) take it to be exogenous (as I do) and call it an "external characteristic."

[28] This feasibility condition is straightforward to understand when each agent is one of a kind. In the general case, feasibility requires that there be, for each group $G \subseteq N$ of a feasible partnership-kind, a common amount of partnership G that its members consume.

[29] A useful implication of this restriction on prices is that being alone is free. That is, for each $i \in N$, $p_{\kappa_i \{\kappa_i\}} = 0$. This comes from letting $T = \{\kappa_i\}$ in $\sum_{\kappa \in T} p_{\kappa T} = 0$.

E.3. Existence of a DIP equilibrium

Theorem 3. *Every economy where each agent is endowed with a positive amount of money has a DIP equilibrium.*

Proof. Let $\bar{\mathbb{P}} \equiv \{p \in \mathbb{P} : ||p|| \leq 1\}$. Let $(u, \omega) \in \mathcal{E}$ be such that for each $i \in N$, $\omega_i > 0$. Define $F_i : \bar{\mathbb{P}} \rightrightarrows X_i$ by setting for each $p \in \bar{\mathbb{P}}$,

$$F_i(p) \equiv \left\{ (\pi_i, x_i) \in X_i : \begin{array}{c} \pi_i \cdot p_{\kappa_i} + x_i p_0 \leq \omega_i p_0 + \frac{1-||p||}{n} \\ \text{and for each } (\pi_i', x_i') \text{ such that} \\ \pi_i' \cdot p_{\kappa_i} + x_i' p_0 < \omega_i p_0 + \frac{1-||p||}{n} \\ u_i(\pi_i, x_i) \geq u_i(\pi_i', x_i') \end{array} \right\}.$$

For each $(\pi, x) \in X$, $d_0(\pi, x) \equiv \sum_{i \in N} x_i - \sum_{i \in N} \omega_i$. For each $\kappa \in K$ and each $T \in \mathcal{S}_\kappa$,

$$d_{\kappa T}(\pi, x) \equiv \left(\sum_{i \in N_\kappa} \pi_{iT} \right) - \frac{\left(\sum_{\theta \in T} \sum_{i \in N_\theta} \pi_{iT} \right)}{|T|}$$

That is, $d_{\kappa T}(\pi, x)$ is the difference between the total amount of T demanded by agents of kind κ and the average amount demanded by all of the kinds that are in T. It is immediate from this definition that for each $(\pi, x) \in X$, for each $T \in \mathcal{S}$, $\sum_{\kappa \in T} d_{\kappa T}(\pi, x) = 0$ if and only if for each pair $\kappa, \theta \in T$, $\sum_{i \in N_\kappa} \pi_{iT} = \sum_{i \in N_\theta} \pi_{iT}$.

If $d(\pi, x) \neq 0$ then $\frac{d(\pi, x)}{||d(\pi, x)||} \in \bar{\mathbb{P}}$. So, define $F_0(\pi, x)$ as follows:

$$F_0(\pi, x) \equiv \begin{cases} \left\{ \frac{d(\pi, x)}{||d(\pi, x)||} \right\} & \text{if } d(\pi, x) \neq 0 \text{ or} \\ \bar{\mathbb{P}} & \text{otherwise.} \end{cases}$$

Define $F : X \times \bar{\mathbb{P}} \rightrightarrows X \times \bar{\mathbb{P}}$ by setting for each $((\pi, x), p) \in X \times \bar{\mathbb{P}}$,

$$F((\pi, x), p) \equiv \underset{i \in N}{\times} F_i(p) \times F_0(\pi, x).$$

The proof that for each $i \in N$, F_i is nonempty, closed, and convex valued and upper hemicontinuous is exactly as in the proof of Theorem 1.

By Kakutani's fixed point theorem, F has a fixed point $((\pi, x), p)$. I establish that this is a DIP equilibrium of (u, ω) in four steps.

Step 1. Show that $||p|| = 1$: I first establish that for each $i \in N$, i's relaxed budget constraint binds at (π_i, x_i). Suppose that $\pi_i \cdot p_{\kappa_i} + x_i p_0 < \omega_i p_0 + \frac{1-||p||}{n}$. If $p_0 = 0$, for each $\alpha \in \mathbb{R}_+$, such that $\alpha > 0$, $\pi_i \cdot p_{\kappa_i} + (x_i + \alpha) p_0 = \pi_i \cdot p_{\kappa_i} + x_i p_0 < \omega_i p_0 + \frac{1-||p||}{n}$. Since u_i is monotonic in money, $u_i(\pi_i, x_i + \alpha) > u_i(\pi_i, x_i)$. This contradicts $(\pi_i, x_i) \in F_i(p)$. If $p_0 > 0$, let $\alpha \in \mathbb{R}_+$ be such that $0 < \alpha < \frac{\frac{1-||p||}{n} + \omega_i p_0 - (\pi_i \cdot p_{\kappa_i} + x_i p_0)}{p_0}$. Then, $\pi_i \cdot p_{\kappa_i} + (x_i + \alpha) p_0 < \omega_i p_0 + \frac{1-||p||}{n}$. Since u_i is monotonic in money, $u_i(\pi_i, x_i + \alpha) > u_i(\pi_i, x_i)$. This contradicts $(\pi_i, x_i) \in F_i(p)$.

Thus, for each $i \in N$,

$$\pi_i \cdot p_{\kappa_i} + x_i p_0 = \omega_i p_0 + \frac{1 - ||p||}{n}. \tag{5}$$

If $||p|| \neq 1$, by definition of F_0, $d(\pi, x) = 0$.

Summing (5) over all $i \in N$,

$$\sum_{i \in N} (\pi_i \cdot p_{\kappa_i} + x_i p_0) = \sum_{i \in N} \omega_i p_0 + 1 - ||p||.$$

Or

$$\sum_{i \in N} \sum_{T \in \mathcal{S}_\kappa} \pi_{iT} p_{\kappa_i T} + \left(\sum_{i \in N} x_i - \sum_{i \in N} \omega_i \right) p_0 = 1 - ||p||.$$

Since $d_0(\pi, x) = 0$, $\sum_{i \in N} x_i - \sum_{i \in N} \omega_i = 0$. Thus,

$$\sum_{\kappa\in K}\sum_{T\in\mathcal{S}_\kappa}\sum_{i\in N_\kappa}\pi_{iT}p_{\kappa T}=1-||p||.$$

That is,

$$\sum_{T\in\mathcal{S}}\sum_{\kappa\in T}p_{\kappa T}\sum_{i\in N_\kappa}\pi_{iT}=1-||p||.$$

Since for each $T\in\mathcal{S}$ and each $\kappa\in T$, $d_{\kappa T}(\pi,x)=0$, every kind in T collectively demands the same amount, π_T, of T. That is, for each $\kappa\in T$, $\sum_{i\in N_\kappa}\pi_{iT}=\pi_T$. So,

$$\sum_{T\in\mathcal{S}}\sum_{\kappa\in T}p_{\kappa T}\pi_T=1-||p||=\sum_{T\in\mathcal{S}}\pi_T\sum_{\kappa\in T}p_{\kappa T}.$$

However, since $p\in\bar{\mathbb{P}}$, for each $T\in\mathcal{S}$, $\sum_{\kappa\in T}p_{\kappa T}=0$. Thus, $1-||p||=0$. This contradicts the assumption that $||p||<1$.

Step 2. Show that $d(\pi,x)=0$: If not, by definition of F_0, $p=\frac{d(\pi,x)}{||d(\pi,x)||}$. Then,

$$d(\pi,x)\cdot p=d(\pi,x)\cdot\frac{d(\pi,x)}{||d(\pi,x)||}=||d(\pi,x)||>0.$$

By Step 1, $||p||=1$. Thus, for each $i\in N$,

$$\pi_i\cdot p_{\kappa_i}+x_ip_0\le\omega_ip_0. \tag{6}$$

Summing (6) over all $i\in N$,

$$\sum_{i\in N}\left(\pi_i\cdot p_{\kappa_i}+x_ip_0\right)\le\sum_{i\in N}\omega_ip_0.$$

That is,

$$\sum_{i\in N}\sum_{T\in\mathcal{S}_{\kappa_i}}\pi_{iT}p_{\kappa_iT}+\sum_{i\in N}(x_i-\omega_i)p_0\le 0.$$

So

$$\sum_{\kappa\in K}\sum_{T\in\mathcal{S}_\kappa}p_{\kappa T}\sum_{i\in N_k}\pi_{iT}+d_0(\pi,x)p_0\le 0.$$

For each $T\in\mathcal{S}$, let $\overline{\pi}_T$ be the average demand for T by its member kinds. That is,

$$\overline{\pi}_T=\frac{\sum_{\kappa\in T}\sum_{i\in N_\kappa}\pi_{iT}}{|T|}.$$

Then, for each $\kappa\in T$, $d_{\kappa T}(\pi,x)=\sum_{i\in N_\kappa}\pi_{iT}-\overline{\pi}_T$. Thus,

$$\sum_{\kappa\in K}\sum_{T\in\mathcal{S}_\kappa}\left[p_{\kappa T}\sum_{i\in N_k}\pi_{iT}-\overline{\pi}_Tp_{\kappa T}+\overline{\pi}_Tp_{\kappa T}\right]+d_0(\pi,x)p_0\le 0.$$

So

$$\sum_{T\in\mathcal{S}}\sum_{\kappa\in T}\left[\sum_{i\in N_\kappa}\pi_{iT}-\overline{\pi}_T\right]p_{\kappa T}+\sum_{T\in\mathcal{S}}\sum_{\kappa\in T}\overline{\pi}_{\kappa T}p_{\kappa T}+d_0(\pi,x)p_0\le 0.$$

That is,

$$\sum_{T\in\mathcal{S}}\sum_{\kappa\in T}d_{\kappa T}(\pi,x)p_{\kappa T}+d_0(\pi,x)p_0+\sum_{T\in\mathcal{S}}\overline{\pi}_T\sum_{\kappa\in T}p_{\kappa T}\le 0.$$

Since $p \in \bar{\mathbb{P}}$, for each $T \in \mathcal{S}$, $\sum_{\kappa \in T} p_{\kappa T} = 0$. Thus, $d(\pi, x) \cdot p \leq 0$. However, this contradicts the earlier assertion that $d(\pi, x) \cdot p > 0$.

The proofs of Step 3 (that $p_0 > 0$) and Step 4 (that for each $i \in N$, (π_i, x_i) is maximal in $B_i(p_i, \omega_i)$) are exactly the same as in the proof of Theorem 1. □

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