J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Projection operators nearly orthogonal to their symmetries ☆

Sam Walters

Department of Mathematics & Statistics, University of Northern B.C., Prince George, B.C. V2N 4Z9, Canada

ARTICLE INFO

Article history:
Received 28 June 2016
Available online xxxx
Submitted by H. Lin

Keywords:
Hilbert space
Norm approximation
C*-algebras
Automorphisms
Projections
Orthogonality

ABSTRACT

For any order 2 automorphism α of a C*-algebra A (a symmetry of A), we prove that for each projection e such that $\|e\alpha(e)\| \le \frac{9}{20}$, there exists a projection q with $q\alpha(q) = 0$ satisfying the norm estimate

$$\|e - q\| \le \frac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2.$$

In other words, if e is a projection that is "nearly orthogonal" to its symmetry $\alpha(e)$ in the sense that the norm $\|e\alpha(e)\|$ is no more than $\frac{9}{20}$, then e can be approximated by a projection q that is exactly orthogonal to its symmetry in a fairly optimal fashion. (Optimal in the sense that the first term in the estimate satisfies $\frac{1}{2}\|e\alpha(e)\| \le \|e-q\|$ for any such q.)

1. Introduction

The purpose of this paper is to obtain a fine estimate for the norm difference $\|e - q\|$ in terms of the norm $\|e\alpha(e)\|$ of a projection e relative to a symmetry α (order 2 automorphism), where q is a projection that is orthogonal to its symmetry (i.e. $q\alpha(q) = 0$). The norm $\|e\alpha(e)\|$ measures the degree to which e is or is not orthogonal to its symmetric image $\alpha(e)$. It is shown that for all C*-algebras this degree does not have to be too small in order that the projection e can be approximated by a projection q that is exactly orthogonal to its symmetry. We show the existence for such fine approximation when the norm $\|e\alpha(e)\|$ is at most $\frac{9}{20} = 0.45$. Further, a bound for the norm $\|e - q\|$ is expressed in terms of a simple quadratic function of $\|e\alpha(e)\|$. The main result is the following.

☆ Research partly supported by a grant from NSERC.
E-mail address: walters@unbc.ca.
URL: http://hilbert.unbc.ca.

http://dx.doi.org/10.1016/j.jmaa.2016.09.013

Theorem 1.1. *Let A be any C^*-algebra and α a symmetry of A. If e is a projection in A such that $\|e\alpha(e)\| < \xi (\approx 0.455)$, then there exists a projection q in the C^*-subalgebra generated by $e, \alpha(e)$ such that*

$$q\alpha(q) = 0, \qquad \|e - q\| \leq \tfrac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2. \tag{1.1}$$

Theorem 1.2. *Let e be any projection operator and u any Hermitian unitary operator on Hilbert space such that $\|eue\| < \xi(\approx 0.455)$. Then there exists a projection operator q such that*

$$quq = 0, \qquad \|e - q\| \leq \tfrac{1}{2}\|eue\| + 4\|eue\|^2.$$

Further, q is in the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by e, ueu^*.

The number $\xi \approx 0.4550898$ is the positive root of $x^2(2+4F(x^2)) = 1$ (where F is defined by (2.1) below). It is clear that Theorem 1.2 follows from 1.1 (since the symmetry on $\mathcal{B}(\mathcal{H})$ in this case is $\alpha(x) = uxu^*$).

The precision of the inequality (1.1) is recognized by noting that the norm $\|e - q\|$ is always at least the first term on the right side:

$$\tfrac{1}{2}\|e\alpha(e)\| \leq \|e - q\| \tag{1.2}$$

for any projection q that is orthogonal to its symmetry ($q\alpha(q) = 0$). Indeed, this is easy to see from the equality

$$e\alpha(e) = (e - q)\alpha(q) + e\alpha(e - q)$$

which gives (1.2). The theorem therefore estimates the norm $\|e-q\|$ from its minimum value (over such q's) to within a quadratic order of magnitude:

$$\tfrac{1}{2}\|e\alpha(e)\| \;\leq\; \|e - q\| \;\leq\; \tfrac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2.$$

In order to improve our estimates, we used the following anticommutator norm formula that we proved in [2].

Theorem 1.3. *(See [2].) For any two projection operators f, g on Hilbert space, one has*

$$\|fg + gf\| = \|fg\| + \|fg\|^2.$$

We note that a C*-algebra A that possesses a symmetry contains non-trivial α-orthogonal positive elements. For example, pick a Hermitian element h such that $\alpha(h) \neq h$ and let $x = h - \alpha(h)$, a nonzero Hermitian element such that $\alpha(x) = -x$. The positive part $a = \frac{1}{2}(|x| + x)$ of x is non-zero (since the spectrum of x contains positive and negative real numbers) and clearly satisfies $a\alpha(a) = 0$. If further, the hereditary C*-subalgebra generated by a, namely $\overline{aAa}$, contains projections then these will automatically be α-orthogonal projections. In particular, if A has real rank zero[1] and has a symmetry, then it contains many α-orthogonal projections.

Theorem 1.1 can be applied in particular to the flip automorphism $U \to U^{-1}, V \to V^{-1}$ of the rotation C*-algebra A_θ generated by unitaries U, V subject to the commutation relation $VU = e^{2\pi i\theta}UV$ – or, indeed, to the flip on any higher dimensional noncommutative torus. The result can also be applied to the noncommutative Fourier transform $U \to V \to U^{-1}$ restricted the fixed point subalgebra of A_θ under the flip.

[1] That is, each Hermitian element can be approximated by a Hermitian with finite spectrum.

Our work is somewhat related to some results in [1] concerning semiprojective group actions (G, A, α), except that our C*-algebra A is not assumed to be semiprojective, nor the action, and that we give a new and precise quantitative result for the action (which is not dealt with in [1]).

2. Proof of Main Theorem

Write $\chi = \chi_{[\frac{1}{2},\infty)}$ for the characteristic function of $[\frac{1}{2}, \infty)$. The following lemma is a slightly finer version of a well-known result, and for the sake of being complete we detailed its proof in the Appendix. We denote by $C^*(x, y)$ the C*-subalgebra generated by elements x, y. We consider the function of a real variable

$$F(x) = \frac{1}{1 + \sqrt{1 - 4x}} \tag{2.1}$$

which is increasing for $0 \le x \le \frac{1}{4}$ and has range $[\frac{1}{2}, 1]$.

Lemma 2.1. *Let A be a C^*-algebra and $a \in A$ be a Hermitian element such that $\delta := \|a^2 - a\| < \frac{1}{4}$. Then the projection $p = \chi(a)$ exists in $C^*(a)$ and satisfies*

$$\|a - p\| \le 2\delta F(\delta).$$

Lemma 2.2. *The inequality $\|b - 1\| \le \|b^2 - 1\|$ holds for any positive element b in a unital C^*-algebra.*

Proof. Since $1 \le (b + 1)^2$ and $b - 1$ is Hermitian, we get

$$(b - 1)1(b - 1) \le (b - 1)(b + 1)^2(b - 1),$$

or $0 \le (b - 1)^2 \le (b^2 - 1)^2$. Taking norms gives

$$\|b - 1\|^2 = \|(b - 1)^2\| \le \|(b^2 - 1)^2\| = \|b^2 - 1\|^2$$

hence the result. □

Proposition 2.3. *Let f and g be projections in a C^*-algebra A such that $t := \|fg\| < \frac{1}{2}$. Then the projection $p = \chi(y^2)$ exists, where $y = g - f$, and has the following properties:*

$$g \le p, \quad f \le p, \quad yp = y \tag{2.2}$$

$$\|y^4 - y^2\| \le t^2 \tag{2.3}$$

$$\|y^2 - p\| \le 2t^2 F(t^2) \tag{2.4}$$

$$\|y^3 - y\| \le 2t^2 F(t^2) \tag{2.5}$$

$$\||y| - p\| \le 2t^2 F(t^2) \tag{2.6}$$

$$\|(f + g) - p\| \le t + t^2 + 2t^2 F(t^2) \tag{2.7}$$

Proof. Let $b = g + f$, a positive element, and let $y = g - f$, a Hermitian element with $-1 \le y \le 1$. We have

$$y^2 = g + f - gf - fg$$

(a positive element with $|y| \le 1$) and

$$\begin{aligned} y^4 &= (g+f-gf-fg)(g+f-gf-fg) \\ &= g+gf-gf-gfg+fg+f-fgf-fg \\ &\quad -gfg-gf+gfgf+gfg-fg-fgf+fgf+fgfg \\ &= g+f-gf-fg-gfg+gfgf-fgf+fgfg. \end{aligned}$$

Thus

$$y^2-y^4 = gfg-gfgf+fgf-fgfg.$$

This can be written as the sum of two orthogonal positive elements

$$y^2-y^4 = u^*u+uu^*$$

where $u = fg - fgf$ satisfies $u^2 = 0$. The norm of $y^2 - y^4$ is therefore just $\|u^*u\| = \|u\|^2$. Since $\|u\| \le \|fg\| < \frac{1}{2}$, we get[2]

$$\|y^2-y^4\| \le \|fg\|^2 = t^2 < \tfrac{1}{4}$$

which in particular gives (2.3). Lemma 2.1 therefore gives the existence of the spectral projection $p := \chi(y^2)$ satisfying

$$\|y^2-p\| \le 2t^2F(t^2)$$

which gives (2.4). Note that y^2 commutes with g and f, so that p also commutes with g and f. Thus pg is a projection under g and p.

It is easy to see that

$$g(y^2-p)g+gfg = g-pg,$$

hence

$$\|g-pg\| \le \|g(y^2-p)g\| + \|fg\|^2 \ \le 2t^2F(t^2)+t^2 \le 3t^2 < 1.$$

Therefore g and pg are equivalent projections in the commutative C*-algebra generated by them, so they must be equal $pg = g$; hence $g \le p$. In exactly the same way, one shows $f \le p$. In particular, p acts as the identity for y: $py = yp = y$. This establishes (2.2).

From Theorem 1.3 and the above estimates one gets

$$\|(f+g)-p\| \le \|(f+g)-y^2\| + \|y^2-p\| = \|fg+gf\| + \|y^2-p\| \tag{2.8}$$

$$\le \|fg\| + \|fg\|^2 + 2t^2F(t^2) \tag{2.9}$$

giving (2.7).

Since p is the identity for y, Lemma 2.2 gives inequality (2.6):

$$\||y|-p\| \le \|y^2-p\| \le 2t^2F(t^2).$$

Also, $\|y^3-y\| = \|y(y^2-p)\| \le \|y^2-p\|$ which gives (2.5), since $\|y\| \le 1$. $\square$

---

[2] We point out that norm differences of higher powers of y such as $\|y^4-y^8\|, \|y^6-y^{12}\|$ can be greater than $\|y^2-y^4\|$, as can be shown for 2 by 2 matrices.

We now proceed to prove the main theorem.

Theorem 2.4. *Let A be any C^*-algebra and α a symmetry of A. If e is a projection in A such that $\|e\alpha(e)\| < \xi(\approx 0.455)$, then there exists a projection q in $C^*(e, \alpha(e))$ such that*

$$q\alpha(q) = 0, \qquad \|e - q\| \le \tfrac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2.$$

Proof. As in the notation of the proof of Proposition 2.3, with $g = e$, $f = \alpha(e)$ we let $y = e - \alpha(e)$ denote the Hermitian element such that $\alpha(y) = -y$ (and $\|y\| \le 1$), and $p = \chi(y^2)$ the associated projection satisfying the properties listed in the proposition. (In particular, we have $py = y$.)

Let $t = \|e\alpha(e)\|$, and let $d = \frac{1}{2}(|y| + y) \in C^*(e, \alpha(e))$ denote the positive part of y. One has $|y| = d + \alpha(d)$, $y = d - \alpha(d)$, and

$$d\alpha(d) = \tfrac{1}{4}(|y| + y)(|y| - y) = \tfrac{1}{4}(|y|^2 - y^2) = 0$$

so that d is an α-orthogonal positive element. One checks that

$$d^2 = \tfrac{1}{2}(y^2 + y|y|), \quad d^4 = \tfrac{1}{2}(y^4 + y^3|y|),$$

whence from inequalities (2.3) and (2.5) one gets

$$\begin{aligned}\|d^4 - d^2\| &= \tfrac{1}{2}\Big\|(y^4 - y^2) + (y^3 - y)|y|\Big\| \\ &\le \tfrac{1}{2}\|y^4 - y^2\| + \tfrac{1}{2}\|y^3 - y\| \le \tfrac{1}{2}t^2 + t^2F(t^2).\end{aligned}$$

With $\xi(= 0.455\cdots)$ being the positive root of the equation $\frac{1}{2}t^2 + t^2F(t^2) = \frac{1}{4}$, we have shown that $\|d^4 - d^2\| < \frac{1}{4}$ when $t < \xi$. Therefore, Lemma 2.1 gives the projection $q = \chi(d^2)$ satisfying $q\alpha(q) = 0$ (since $d\alpha(d) = 0$), and

$$\|q - d^2\| \le t^2(1 + 2F(t^2))F(\tfrac{1}{2}t^2 + t^2F(t^2)).$$

Let us compute

$$\begin{aligned}2e - 2d^2 &= 2e - (y^2 + y|y|) = 2e - [(e - \alpha(e))^2 + y|y|] \\ &= 2e - [e + \alpha(e) - e\alpha(e) - \alpha(e)e + y|y|] \\ &= e - \alpha(e) + e\alpha(e) + \alpha(e)e - y|y| \\ &= y - y|y| + e\alpha(e) + \alpha(e)e \\ &= y(p - |y|) + e\alpha(e) + \alpha(e)e\end{aligned}$$

where p is the projection of Proposition 2.3 (since $\|e\alpha(e)\| < \frac{1}{2}$ already holds). Using inequality (2.6) and Theorem 1.3 we get

$$\|e - d^2\| \le \tfrac{1}{2}t + \tfrac{1}{2}t^2 + t^2F(t^2).$$

Thus,

$$\begin{aligned}\|e - q\| &\le \|e - d^2\| + \|d^2 - q\| \\ &\le \tfrac{1}{2}t + \tfrac{1}{2}t^2 + t^2F(t^2) + t^2(1 + 2F(t^2))F(\tfrac{1}{2}t^2 + t^2F(t^2)) && (2.10) \\ &= \tfrac{1}{2}t + t^2G(t) && (2.11)\end{aligned}$$

Please cite this article in press as: S. Walters, Projection operators nearly orthogonal to their symmetries, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.09.013

where

$$G(t) = \tfrac{1}{2} + F(t^2) + (1 + 2F(t^2))F(\tfrac{1}{2}t^2 + t^2F(t^2)).$$

Since G is an increasing function (recall that F is increasing), we can replace it by its maximum over $[0, \xi]$, namely $G(\xi) = 3.62... < 4$, and we therefore obtain

$$\|e - q\| \le \tfrac{1}{2}t + t^2G(t) \le \tfrac{1}{2}t + 4t^2.$$

This completes the proof. □

3. Appendix

Lemma 3.1. *Let A be a C^*-algebra and $a \in A$ be a Hermitian element such that $\|a^2 - a\| < \frac{1}{4}$. Then the projection $p = \chi(a)$ exists in $C^*(a)$ and satisfies*

$$\|a - p\| \le 2\|a^2 - a\|F(\|a^2 - a\|).$$

Recall that $\chi = \chi_{[\frac{1}{2},\infty)}$ is the characteristic function of $[\frac{1}{2}, \infty)$, and we note that the projection p does not in general act like an identity for a.

Proof. Let $\delta = \|a^2 - a\|$, so that the spectrum $\mathrm{Sp}(a^2 - a)$ of $a^2 - a$ is contained in the interval $[-\delta, \delta]$. Let $f(x) = x - x^2$. Then by the spectral mapping theorem

$$f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a)) = \mathrm{Sp}(a - a^2) \subseteq [-\delta, \delta].$$

As $\delta < \frac{1}{4}$, we have $\frac{1}{2} \notin \mathrm{Sp}(a)$ so that the projection $p = \chi(a)$ exists in $C^*(a)$. Now if $t \in \mathrm{Sp}(a)$, then $-\delta \le f(t) \le \delta$ whose solutions are easily checked to be one of the intervals

$$\tfrac{1}{2}(1 - \sqrt{1 + 4\delta}) < t < \tfrac{1}{2}(1 - \sqrt{1 - 4\delta}), \tag{3.1}$$

$$\tfrac{1}{2}(1 + \sqrt{1 - 4\delta}) < t < \tfrac{1}{2}(1 + \sqrt{1 + 4\delta}). \tag{3.2}$$

Therefore, if we let $g(x) = x$, then $\|a - p\| = \|g(a) - \chi(a)\| = \|g - \chi\|_{\mathrm{Sp}(a)}$. For t in the first interval (3.1) we have $|g(t) - \chi(t)| = |t|$, and for t the second interval (3.2) we have $|g(t) - \chi(t)| = |t - 1|$. It can be checked that in either case we have $|g(t) - \chi(t)| < \frac{1}{2}(1 - \sqrt{1 - 4\delta})$. Therefore,

$$\|a - p\| \le \tfrac{1}{2}(1 - \sqrt{1 - 4\delta}) = \frac{2\delta}{1 + \sqrt{1 - 4\delta}} = 2\delta F(\delta),$$

as desired. □

Acknowledgments

This research was partly supported by a grant from NSERC (Natural Science and Engineering Council of Canada).

References

[1] N.C. Phillips, A. Sorensen, H. Thiel, Semiprojectivity with and without a group action, J. Funct. Anal. 268 (4) (2015) 929–973, ArXiv.org/abs/1403.3440.

[2] S. Walters, Anticommutator norm formula for projection operators, preprint, ArXiv.org/abs/1604.00699, 2015, 7 pages.