Numer Algor (2007) 44:173–184
DOI 10.1007/s11075-007-9097-z

ORIGINAL PAPER

An efficient algorithm for the generalized centro-symmetric solution of matrix equation $AXB = C$

Mao-lin Liang · Chuan-hua You · Li-fang Dai

Received: 31 December 2006 / Accepted: 13 April 2007 /
Published online: 26 May 2007

Abstract In this paper, an iterative algorithm is constructed for solving linear matrix equation $AXB = C$ over generalized centro-symmetric matrix X. We show that, by this algorithm, a solution or the least-norm solution of the matrix equation $AXB = C$ can be obtained within finite iteration steps in the absence of roundoff errors; we also obtain the optimal approximation solution to a given matrix X_0 in the solution set of which. In addition, given numerical examples show that the iterative method is efficient.

Keywords Algorithm · Generalized centro-symmetric solution · Least-norm solution · Optimal approximation

1 Introduction

We first introduce some notations to be used. Let $R^{m\times n}$ be the set of all $m \times n$ real matrices, $R^m = R^{m\times 1}$, $SR^{n\times n}$ be the set of all symmetric matrices in $R^{n\times n}$, $ASR^{n\times n}$ be the set of all anti-symmetric matrices in $R^{n\times n}$, $SOR^{n\times n}$ be the set of all symmetric orthogonal matrices in $R^{n\times n}$. Denoted by the superscripts T and I_n be the transpose and identity matrix with order n, respectively. For matrices $A = (a_1, a_2, \ldots, a_n)$, $B \in R^{m\times n}$, $a_i \in R^m$, $R(A)$ and $tr(A)$ represent its column space and trace, respectively; Symbol $vec(\cdot)$ represents the vec

M.-l. Liang (✉) · C.-h. You · L.-f. Dai
School of Mathematics and Statistics, Lanzhou University,
Lanzhou, Gansu 730000, People's Republic of China
e-mail: liangml2005@163.com

M.-l. Liang · L.-f. Dai
College of Mathematics–Physics and Information Science, Tianshui Normal University,
Tianshui, Gansu 741001, People's Republic of China

operator, i.e., $vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$; $A \otimes B$ stands for the Kronecker product of matrices A and B; Moreover, $\langle A, B\rangle = tr(B^T A)$ is defined as the inner product of the two matrices, which generates the Frobenius norm, i.e. $\|A\| = \sqrt{< A, A >} = \sqrt{tr(A^T A)}$.

Let $J = (e_n, e_{n-1}, \ldots, e_2, e_1)$, e_i is the ith column of identity matrix I_n, noting that $J = J^T = J^{-1}$. If $JAJ = A$, we say that $A \in R^{n\times n}$ is centro-symmetric matrix, which has practical applications in information theory, linear system theory, linear estimate theory and numerical analysis (see [1, 2]). As the extension of centro-symmetric matrix, we define the following conception (see [10] for detail).

Definition 1.1 For arbitrary given matrix $P \in SOR^{n\times n}$, i.e., $P = P^T = P^{-1}$, we say that matrix $A \in R^{n\times n}$ is generalized centro-symmetric (generalized central anti-symmetric) with respect to P, if $PAP = A$ ($PAP = -A$). The set of order n generalized centro-symmetric (generalized central anti-symmetric) matrices with respect to P is denoted by $CSR_P^{n\times n}$ ($CASR_P^{n\times n}$).

In addition, the following definition is necessary.

Definition 1.2 Assume $M, N \in R^{s\times t}$, where s, t are arbitrary positive integers, if $tr(M^T N) = 0$, matrices M, N are called orthogonal each other.

Remark 1 In this paper, let the given symmetric orthogonal matrix P be the same as in Definition 1.1.

From Definitions 1.1 and 1.2, if matrices $F \in CSR_P^{n\times n}$, $G \in CASR_P^{n\times n}$, it is easy to verify that $tr(F^T G) = 0$, i.e., F, G are orthogonal each other, then for given matrix P, we have the following lemma.

Lemma 1.1 $R^{n\times n} = CSR_P^{n\times n} \oplus CASR_P^{n\times n} \oplus ASR^{n\times n}$.

We consider the following two problems.

Problem I For given matrices $P \in SOR^{n\times n}$, $A \in R^{m\times n}$, $B \in R^{n\times p}$ and $C \in R^{m\times p}$, find $X \in CSR_P^{n\times n}$, such that

$$AXB = C. \tag{1}$$

Problem II When Problem I is consistent, let S_E denote the solution set of (1), for given matrix $X_0 \in R^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\|\hat{X} - X_0\| = \min_{X\in S_E} \|X - X_0\| \tag{2}$$

In fact, (2) is to find the nearness matrix to X_0 in S_E.

Linear matrix equation (1) has been discussed extensively with some special unknown X, such as, Dai [3] and Chu [4] has studied the symmetric X by singular value decomposition (SVD); Mitra [5] has obtain the common solution of simultaneous matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ by *g*-inverse; And the reflexive and anti-reflexive solutions of matrix equation (1) have been represented in [6], by generalized SVD. In addition, Pengs [7–9] have constructed iterative methods to find the symmetric solutions and least-squares solutions of the matrix equation (1) and matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, respectively. For generalized centro-symmetric matrix, the inverse eigenvalue problem has been considered by Xie in [10].

However, the generalized centro-symmetric solution of matrix equation $AXB = C$ has not been studied by iterative method. In this paper, we establish an iterative algorithm, which is similar but different from that in reference [7], to solve Problems I and II.

This paper is organized as follows. An iterative algorithm will be constructed to solve Problem I in Section 2. We will show that, for any initial matrix X_1, a solution or the least-norm solution of matrix equation (1) can be obtained within finite iterative steps in the absence of roundoff errors; In Section 3, applying to this iterative method, we will derive the optimal approximation solution to a given matrix X_0 by a new matrix equation $A\tilde{X}B = \tilde{C}$, where $\tilde{X} = X - X_0$, $\tilde{C} = C - CX_0B$, that is Problem II; in Section 4, we will offer some numerical examples to verify our results.

2 An iterative method for solving Problem I

In this section, we will establish an iterative algorithm to solve Problem I, some lemmas will be given to analyze the properties of the algorithm. Meanwhile, we will show that any solution of the matrix equation (1) can be obtained within finite iterative steps.

Algorithm 2.1

Step 1 : Input matrices $A \in R^{m\times n}$, $B \in R^{n\times p}$, $C \in R^{m\times p}$, $P \in SOR^{n\times n}$. Choose an arbitrary matrix $X_1 \in CSR_P^{n\times n}$.

Step 2 : Calculate

$$\begin{aligned} R_1 &= C - AX_1B, \\ P_1 &= \frac{1}{2}(A^TR_1B^T + PA^TR_1B^TP), \\ k &:= 1. \end{aligned}$$

Step 3 : Calculate

$$X_{k+1} = X_k + \frac{\| R_k \|^2}{\| P_k \|^2} P_k.$$

Step 4 : Calculate

$$R_{k+1} = C - AX_{k+1}B = R_k - \frac{\| R_k \|^2}{\| P_k \|^2} AP_k B$$

$$P_{k+1} = \frac{1}{2}(A^T R_{k+1} B^T + PA^T R_{k+1} B^T P) + \frac{\| R_{k+1} \|^2}{\| R_k \|^2} P_k.$$

If $R_{k+1} = 0$, or $R_{k+1} \neq 0$, $P_{k+1} = 0$, stop, otherwise, let $k := k + 1$, go to Step 3.

Obviously, we know that X_i, $Q_i \in CSR_P^{n\times n}$ $(i = 1, 2, \dots)$. In the sequel, we will illustrate the feasibility of Algorithm 2.1.

Lemma 2.1 *For the sequences* $\{R_i\}$, $\{P_i\}$ $(i = 1, 2, \dots)$, *in the iterative algorithm, we have*

$$tr(R_{i+1}^T R_j) = tr(R_i^T R_j) - \frac{\| R_i \|^2}{\| P_i \|^2} tr(P_i^T P_j) + \frac{\| R_i \|^2 \| R_j \|^2}{\| P_i \|^2 \| R_{j-1} \|^2} tr(P_i^T P_{j-1}). \quad (3)$$

Proof From Lemma 1.1, 2.1 and the Algorithm 2.1, noting that $PP_iP = P_i$, $P \in SOR^{n\times n}$, we can obtain

$$\begin{aligned} tr[(AP_iB)^T R_j] &= tr(P_i^T A^T R_j B^T) \\ &= tr\left[P_i^T \left(\frac{A^T R_j B^T + PA^T R_j B^T P}{2} + \frac{A^T R_j B^T - PA^T R_j B^T P}{2}\right)\right] \\ &= tr\left(P_i^T \frac{A^T R_j B^T + PA^T R_j B^T P}{2}\right) \\ &= tr\left[P_i^T \left(P_j - \frac{\| R_j \|^2}{\| R_{j-1} \|^2} P_{j-1}\right)\right] \\ &= tr(P_i^T P_j) - \frac{\| R_j \|^2}{\| R_{j-1} \|^2} tr(P_i^T P_{j-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} tr(R_{i+1}^T R_j) &= tr\left[\left(R_i - \frac{\| R_i \|^2}{\| P_i \|^2} AP_iB\right)^T R_j\right] \\ &= tr(R_i^T R_j) - \frac{\| R_i \|^2}{\| P_i \|^2} tr[(AP_iB)^T R_j] \\ &= tr(R_i^T R_j) - \frac{\| R_i \|^2}{\| P_i \|^2} tr(P_i^T P_j) + \frac{\| R_i \|^2 \| R_j \|^2}{\| P_i \|^2 \| R_{j-1} \|^2} tr(P_i^T P_{j-1}). \end{aligned}$$

We complete the proof of equality (3). □

Lemma 2.2 *For sequences* $\{R_i\}$, $\{P_i\}$ *generated by the iterative method, we have that*

$$tr(R_i^T R_j) = 0, \quad tr(P_i^T P_j) = 0,\ i, j = 1, 2, \ldots, k\,(k \geq 2),\ i \neq j. \tag{4}$$

Proof We prove the conclusion by induction.

When $k = 2$, from Lemma 1.1 , Algorithm 2.1 and the proof of Lemma 2.1, then

$$\begin{aligned} tr(R_2^T R_1) &= tr(R_1^T R_1) - \frac{\| R_1 \|^2}{\| P_1 \|^2} tr[(AP_1B)^T R_1) \\ &= tr(R_1^T R_1) - \frac{\| R_1 \|^2}{\| P_1 \|^2} tr(P_1^T P_1) = 0. \end{aligned} \tag{5}$$

$$\begin{aligned} tr(P_2^T P_1) &= tr\left[\left(\frac{A^T R_2 B^T + PA^T R_2 B^T P}{2} + \frac{\|R_2\|^2}{\|R_1\|^2} P_1\right)^T P_1\right] \\ &= tr(R_2^T A P_1 B) + \frac{\|R_2\|^2}{\|R_1\|^2} tr(P_1^T P_1) \\ &= tr\left[R_2^T \frac{\|P_1\|^2}{\|R_1\|^2}(R_1 - R_2)\right] + \frac{\|R_2\|^2}{\|R_1\|^2} tr(P_1^T P_1) \\ &= 0, \end{aligned} \tag{6}$$

Assume (4) holds for $k=s$, that is, $tr(R_s^T R_j)=0$, $tr(P_s^T P_j)=0$, $j = 1, 2, \ldots, s-1$, by Lemma 2.1 and the iterative method, similar to the proofs of (5) and (6), we can verify that $tr(R_{s+1}^T R_s) = 0$, and $tr(P_{s+1}^T P_s) = 0$.

Next, we will complete the proof of Lemma 2.2 if the two items $tr(R_{s+1}^T R_j) = 0$, $tr(P_{s+1}^T P_j) = 0$ hold.

In fact, according the assumptions and the iterative method, when $j = 1$, we have that

$$\begin{aligned} tr(R_{s+1}^T R_1) &= tr(R_s^T R_1) - \frac{\|R_s\|^2}{\|P_s\|^2} tr(AP_sB)^T R_1) \\ &= -\frac{\|R_s\|^2}{\|P_s\|^2} tr(AP_sB)^T R_1) \\ &= -\frac{\|R_s\|^2}{\|P_s\|^2} tr(P_s^T A^T R_1 B^T) \\ &= -\frac{\|R_s\|^2}{\|P_s\|^2} tr(P_s^T P_1) \\ &= 0, \end{aligned}$$

and connecting with (3), then

$$
\begin{aligned}
tr(P_{s+1}^T P_1) &= tr\left[\left(\frac{A^T R_{s+1} B^T + P A^T R_{s+1} B^T P}{2} + \frac{\| R_{s+1}\|^2}{\| R_s\|^2} P_s\right)^T P_1\right] \\
&= tr[R_{s+1}^T (A P_1 B)] + \frac{\| R_{s+1}\|^2}{\| R_s\|^2} tr(P_s^T P_1) \\
&= \frac{\| P_1\|^2}{\| R_1\|^2} tr[R_{s+1}^T (R_1 - R_2)] + \frac{\| R_{s+1}\|^2}{\| R_s\|^2} tr(P_s^T P_1) \\
&= \frac{\| P_1\|^2}{\| R_1\|^2} tr[R_{s+1}^T R_2) \\
&= 0.
\end{aligned} \tag{7}
$$

Furthermore, when $2 \leq j \leq s-1$, Lemma 2.1 and the assumptions imply that

$$
tr(R_{s+1}^T R_j) = tr(R_s^T R_j) - \frac{\| R_s\|^2}{\| P_s\|^2} tr(P_s^T P_j) + \frac{\| R_s\|^2 \| R_j\|^2}{\| P_s\|^2 \| R_{j-1}\|^2} tr(P_s^T P_{j-1}) = 0,
$$

similar to the proof of (7), we have $tr(P_{s+1}^T P_j) = 0$. Hence, (4) holds for $k = s+1$.

Such are the proofs of formulas (4). □

Lemma 2.3 *Suppose that $\bar{X}$ be an arbitrary solution of Problem I, then for R_k, P_k generated by the Algorithm* 2.1*, we have*

$$
tr[(\bar{X} - X_k)^T P_k] = \| R_k \|^2, \ k = 1, 2, \ldots. \tag{8}
$$

Proof When $k = 1$, from the Algorithm 2.1 and Lemma 2.2, we have that

$$
\begin{aligned}
tr[(\bar{X} - X_1)^T P_1] &= \frac{1}{2} tr[(\bar{X} - X_1)^T (A^T R_1 B^T + P A^T R_1 B^T P)] \\
&= \frac{1}{2} tr[B R_1^T A (\bar{X} - X_1) + P B R_1^T A P (\bar{X} - X_1)] \\
&= tr[R_1^T A (\bar{X} - X_1) B] \\
&= tr(R_1^T (C - A X_1 B)] \\
&= \| R_1^T \|^2.
\end{aligned}
$$

Assume equality (8) holds for $k = s$, we get

$$
\begin{aligned}
tr[(\bar{X} - X_{s+1})^T P_s] &= tr\left[\left(\bar{X} - X_s - \frac{\| R_s\|^2}{\| P_s\|^2} P_s\right)^T P_s\right] \\
&= tr[(\bar{X} - X_s)^T P_s] - \frac{\| R_s\|^2}{\| P_s\|^2} tr(P_s^T P_s) \\
&= \| R_s\|^2 - \| R_s\|^2 \\
&= 0,
\end{aligned}
$$

moreover,

$$
\begin{aligned}
&tr[(\bar{X} - X_{s+1})^T P_{s+1}] \\
&\quad = tr\left[(\bar{X}-X_{s+1})^T \frac{A^T R_{s+1} B^T + P A^T R_{s+1} B^T P}{2} + \frac{\| R_{s+1}\|^2}{\| R_s\|^2}(\bar{X}-X_{s+1})^T P_s\right] \\
&\quad = \frac{1}{2} tr[B R_{s+1}^T A(\bar{X} - X_{s+1}) + P B R_{s+1}^T A P(\bar{X} - X_{s+1})] \\
&\qquad + \frac{\| R_{s+1}\|^2}{\| R_s\|^2} tr\left[(\bar{X} - X_{s+1})^T P_s\right] \\
&\quad = tr[R_{s+1}^T A(\bar{X} - X_{s+1}) B] \\
&\quad = tr[R_{s+1}^T (C - A X_{s+1} B)] \\
&\quad = \| R_{s+1}\|^2.
\end{aligned}
$$

Therefore, we complete the proof by the principle of induction. □

Theorem 2.1 *When Problem I is consistent, then for an arbitrary initial matrix* $X_1 \in CSR_P^{n\times n}$*, a solution of matrix equation* (1) *can be obtained within finite iterative steps.*

Proof If $R_i \neq 0, i = 1, 2, \ldots, mp$, we get $P_i \neq 0$ from Lemma 2.3, then we can compute X_{mp+1}, R_{mp+1} by Algorithm 2.1, which satisfy that $tr(R_{mp+1}^T R_i) = 0$, $tr(R_i^T R_j) = 0$, where $i, j = 1, 2, \ldots, mp$, $i \neq j$. Hence, the sequence $\{R_i\}$ consists of an orthogonal basis of matrix space $R^{m\times p}$, which implies that $R_{mp+1} = 0$, i.e., X_{mp+1} is a solution of Problem I.

Furthermore, if Problem I is consistent, we can prove that the solution of which can be obtained within almost $t_0 + 1$ iteration steps, where $t_0 = min(mp, n^2)$. In fact, if $n^2 \leq mp$ and $R_i \neq 0, i = 1, 2, \ldots, n^2$, Lemma 3 implies that $P_i \neq 0$, using the iterative algorithm, we can compute X_{n^2+1}, R_{n^2+1}, P_{n^2+1}. Similar to the previous proof, we can obtain $P_{n^2+1} = 0$, and $R_{n^2+1} = 0$, i.e., X_{n^2+1} is a solution of Problem I. □

From Theorem 2.1, we have the following assertion.

Corollary 2.1 *Problem I is inconsistent, if and only if there exists a positive integer k, such that* $R_k \neq 0$ *and* $P_k = 0$ *in the process of the iteration.*

Proof If there exists a positive integer k, such that $R_k \neq 0$ and $P_k = 0$, which contradicts to Lemma 2.3, so Problem I is inconsistent.

Conversely, the inconsistency of Problem I implies that $R_i \neq 0$ for all positive integer i. If $P_i \neq 0$ for all positive integer i, then Problem I has a solution by Theorem 2.1, which contradicts to the inconsistency. Therefore, the conclusion holds. □

Theorem 2.1 and its corollary imply that, in the absence of roundoff errors, the solvability of Problem I can be determined automatically within finite iterative steps.

The following lemma can be seen in [7].

Lemma 2.4 *Suppose that the consistent linear equations $My = b$ has a solution $y_0 \in R(M^T)$, then y_0 is the unique least-norm solution of which.*

Theorem 2.2 *Suppose that Problem I is consistent. Let initial iteration matrix $X_1 = A^T H B^T + P A^T H B^T P$, where arbitrary $H \in R^{m\times p}$, or especially, $X_1 = 0 \in R^{n\times n}$, then the solution, generated by the Algorithm 2.1, is the unique least-norm solution of Problem I.*

Proof Algorithm 2.1 and Theorem 2.1 imply that, if let $X_1 = A^T H B^T + P A^T H B^T P$, where H is an arbitrary matrix in $R^{m\times p}$, we can obtain a solution X^* of Problem I within finite iteration steps, which has form like that $X^* = A^T Y B^T + P A^T Y B^T P$. Hence, in order to prove the conclusion, it is enough to show that X^* is the least-norm of Problem I.

Considering the following matrix equations with $X \in CSR_P^{n\times n}$

$$\begin{cases} AXB = C \\ APXPB = C \end{cases}.$$

Obviously, the solvability of which is equivalent to that of Problem I.

Moreover, denote $vec(X^*) = x^*$, $vec(X) = x$, $vec(Y) = y$, $vec(C) = c$, then the matrix equations can be transformed equivalently to

$$\begin{pmatrix} B^T \otimes A \\ (B^T P) \otimes (AP) \end{pmatrix} x = \begin{pmatrix} c \\ c \end{pmatrix}.$$

In addition, by above notations, X^* can be rewritten as

$$x^* = \begin{pmatrix} B^T \otimes A \\ (B^T P) \otimes (AP) \end{pmatrix}^T \begin{pmatrix} y \\ y \end{pmatrix} \in R\left(\begin{pmatrix} B^T \otimes A \\ (B^T P) \otimes (AP) \end{pmatrix}^T\right),$$

which implies that, from Lemma 2.4, X^* is the least-norm solution of the matrix equations, so is that of Problem I. □

3 The solution of Problem II

Suppose that Problem I is consistent, i.e., S_E is not empty. It is easy to verify that S_E is a closed convex set in $CSR_P^{n\times n}$, so the optimal approximation solution is unique. Without loss of generality, we assume that the given matrix

$X_0 \in CSR_P^{n\times n}$ in Problem II. In fact, from Lemma 1.1, for $X \in S_E$, we have that

$$\begin{aligned}\|X - X_0\|^2 &= \left\| X - X_1 - \frac{X_0 - X_0^T}{2} \right\|^2 \\ &= \left\| X - \frac{X_1 + PX_1^TP}{2} - \frac{X_1 - PX_1^TP}{2} - \frac{X_0 - X_0^T}{2} \right\|^2 \\ &= \left\| X - \frac{X_1 + PX_1^TP}{2} \right\|^2 + \left\| \frac{X_1 - PX_1^TP}{2} \right\|^2 + \left\| \frac{X_0 - X_0^T}{2} \right\|^2,\end{aligned}$$

where $X_1 = \dfrac{X_0 + X_0^T}{2}$.

Writing $\tilde{X} = X - X_0, \tilde{C} = C - AX_0B$, then Problem II is equivalent to find the least-norm solution $\tilde{X}^* \in CSR^{n\times n}$ of the following matrix equation

$$A\tilde{X}B = \tilde{C}. \tag{9}$$

By Theorem 2.2, if let initial iteration matrix $\tilde{X}_1 = A^T\tilde{H}B^T + PA^T\tilde{H}B^TP$, where $\tilde{H}$ is an arbitrary matrix in $R^{m\times p}$, or especially, let $\tilde{X}_1 = 0 \in R^{n\times n}$, we can obtain the unique least-norm solution $\tilde{X}^*$ of matrix equation (9) by the Algorithm 2.1. Furthermore, the unique optimal approximation solution $\hat{X}$ to X_0 can be obtained by $\hat{X} = \tilde{X}^* + X_0$.

4 Several numerical examples

Example 1 Let $m = n = 6$, $p = 5$, choosing matrices A, B, C and arbitrary given symmetric orthogonal matrix P as follows:

$$A = \begin{pmatrix} -3 & -5 & -2 & 2 & 9 & -3 \\ 0 & -4 & 9 & -9 & -2 & -8 \\ 6 & 1 & -7 & 7 & 1 & 4 \\ -2 & -4 & 5 & -5 & -8 & -3 \\ -1 & 6 & -2 & 2 & -2 & 0 \\ 0 & 9 & 1 & -1 & -8 & -6 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 2 & 0 & -5 & -3 \\ 0 & -4 & 6 & 2 & -6 \\ -6 & 1 & -7 & 0 & 5 \\ 2 & 5 & 3 & 8 & -3 \\ 4 & -3 & 1 & -2 & 0 \\ -7 & 5 & -7 & 4 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 48 & 195 & 235 & 241 & -173 \\ -398 & 442 & -168 & 1096 & -104 \\ 562 & -308 & 512 & -576 & -171 \\ -279 & 271 & -239 & 578 & 3 \\ 111 & -98 & -344 & -475 & 273 \\ 120 & 101 & -723 & -232 & 431 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Next, we will find the associated solutions of Problems I and II by Algorithm 2.1. However, R_i will usually unequal to zero in the iterative progress for the influence of roundoff errors, therefore, for any chosen positive number ε, however small enough, e.g., $\varepsilon = 1.0000e-010$, whenever $\|R_k\| < \varepsilon$, stop the iteration, and X_k is regarded to be a required solution.

(I) The solutions of Problem I:
Choosing arbitrary initial matrix $X_1 \in CSR_P^{6\times 6}$, for instance,

$$X_1 = \begin{pmatrix} -15 & 0 & -10 & 0 & 0 & 0 \\ 0 & 20 & 0 & 25 & 16 & 12 \\ -16 & 0 & -27 & 0 & 0 & 0 \\ 0 & -13 & 0 & 10 & 10 & -23 \\ 0 & -12 & 0 & -14 & 25 & -11 \\ 0 & -27 & 0 & -22 & 28 & -13 \end{pmatrix},$$

then, by the iterative method, we obtain a solution of Problem I, that is

$$X_{31}=\begin{pmatrix} -3.0000 & 0 & -8.0000 & 0 & 0 & 0 \\ 0 & -4.0000 & 0 & -5.0000 & 6.0000 & 2.0000 \\ -17.0809 & 0 & -21.0154 & 0 & 0 & 0 \\ 0 & 7.8734 & 0 & 6.8939 & -1.4285 & -14.0877 \\ 0 & 2.0000 & 0 & -0.0000 & -5.0000 & -2.0000 \\ 0 & 4.0000 & 0 & -6.0000 & -8.0000 & -3.0000 \end{pmatrix},$$

and in this case, we have $\|R_{31}\| = 8.4393e-011 < \varepsilon$, $\|X_{31}\| = 36.8161$. In addition, we know that, just as previous analysis, if initial matrix $X_1 = A^T HB^T + PA^T HB^T P$, where arbitrary $H \in R^{6\times 5}$, then the solution X_k generated by the iterative method is the unique least-norm solution X^* of Problem I. Choosing $H \in R^{6\times 5}$ as follows:

$$H=\begin{pmatrix} 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \Rightarrow X_1 = \begin{pmatrix} -216 & 0 & 108 & 0 & 0 & 0 \\ 0 & 528 & 0 & 322 & 66 & -242 \\ -34 & 0 & 176 & 0 & 0 & 0 \\ 0 & 280 & 0 & 316 & -168 & 240 \\ 0 & -68 & 0 & 216 & -132 & 136 \\ 0 & 200 & 0 & 180 & -100 & 212 \end{pmatrix}.$$

By Algorithm 2.1, the least-norm solution X^* is

$$X^* = \bar{X}_{31} = \begin{pmatrix} -3.0000 & 0 & -8.0000 & 0 & 0 & 0 \\ 0 & -4.0000 & 0 & -5.0000 & 6.0000 & 2.0000 \\ -5.8792 & 0 & 2.2509 & 0 & 0 & 0 \\ 0 & -3.1186 & 0 & -2.0970 & 0.0156 & -3.8900 \\ 0 & 2.0000 & 0 & 0.0000 & -5.0000 & -2.0000 \\ 0 & 4.0000 & 0 & -6.0000 & -8.0000 & -3.0000 \end{pmatrix}, \tag{10}$$

at this time, $\|\bar{R}_{31}\| = 9.5866e-011 < \varepsilon$, $\|\bar{X}_{31}\| = 19.5163$.
Especially, if let $X_1 = 0$, we can also obtain the least-norm solution as in (10).

(II) The solution of Problem II
In this part, we will obtain the solution of Problem II by the iterative algorithm in this paper. Suppose that the given matrix $X_0 \in CSR_P^{6\times 6}$ is

$$X_0 = \begin{pmatrix} 5 & 0 & 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 3 & -6 & 2 \\ 3 & 0 & 7 & 0 & 0 & 0 \\ 0 & -3 & 0 & 9 & -7 & -3 \\ 0 & 4 & 0 & 5 & -5 & 8 \\ 0 & 7 & 0 & 2 & -8 & -4 \end{pmatrix}.$$

Computing $C_0 = AX_0B$, then we can obtain the solution $\hat{X}$ of Problem II by finding the least-norm solution $\tilde{X}^*$ of (9), that is,

$$\tilde{X}^* = \begin{pmatrix} -8.0000 & 0 & -14.0000 & 0 & 0 & 0 \\ 0 & -2.0000 & 0 & -8.0000 & 12.0000 & 0.0000 \\ -7.6398 & 0 & -2.1748 & 0 & 0 & 0 \\ 0 & -1.3347 & 0 & -12.0917 & 7.1753 & 0.2383 \\ 0 & -2.0000 & 0 & -5.0000 & 0.0000 & -10.0000 \\ 0 & -3.0000 & 0 & -8.0000 & -0.0000 & 1.0000 \end{pmatrix},$$

furthermore,

$$\hat{X} = \begin{pmatrix} -3.0000 & 0 & -8.0000 & 0 & 0 & 0 \\ 0 & -4.0000 & 0 & -5.0000 & 6.0000 & 2.0000 \\ -4.6398 & 0 & 4.8252 & 0 & 0 & 0 \\ 0 & -4.3347 & 0 & -3.0917 & 0.1753 & -2.7617 \\ 0 & 2.0000 & 0 & 0 & -5.0000 & -2.0000 \\ 0 & 4.0000 & 0 & -6.0000 & -8.0000 & -3.0000 \end{pmatrix}.$$

The following numerical example derives from [7].

Example 2 Suppose that the given matrices

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & -3 & -4 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 \\ 1 & -2 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 20 & 3 & -22 & 2 \\ 24 & 24 & -72 & 6 \\ 16 & -18 & 28 & -2 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Because of the influence of the error of calculation, for any chosen positive number ε, e.g., $\varepsilon = 1.0000e-005$, we will stop the iteration when $\|R_k\| < \varepsilon$, or $\|P_k\| < \varepsilon$. Let initial iterative matrix $X_1 = 0$, by using the iterative method, we obtain that $\|R_7\| = 356.8780 > \varepsilon$, and $\|P_7\| = 7.2120e-006 < \varepsilon$. Therefore, from the Corollary 2.1, we know that Problem I is inconsistent, and has no solution in $CSR_P^{4\times 4}$.

5 Conclusions

In this paper, an iterative algorithm, i.e., Algorithm 2.1, is established to solve matrix equation $AXB = C$ over generalized centro-symmetric matrix X. By this algorithm, the solvability of the equation $AXB = C$ can be determined automatically. When the equation is consistent, for any initial symmetric matrix X_0, its solution can be obtained within finite iterative steps in the absence of roundoff errors; And the least-norm solution of which can be derived by choosing a suitable initial iterative matrix X_1, or especially, let $X_1 = 0$; Furthermore, by the iterative method, the optimal approximation solution to a given matrix X_0 can be obtained from the least-norm generalized centro-symmetric solution of a new matrix equation $A\tilde{X}B = \tilde{C}$. Finally, some numerical examples given have illustrate the efficiency of the iterative method.

Acknowledgements The authors express their gratitude to Prof. Claude Brezinski and the anonymous referee for their significant suggestions.

References

1. Golub, G.H., Van Loan, C.F.: Matrix computations [M]. John Hopkins University Press, Baltimore, MD (1996)
2. Zhou, F.Z., Hu, X.Y., Zhang, L.: The solvability conditions for the inverse eigenvalue problems of centro-symmetric matrices. Linear Algebra Appl. **364**, 147–160 (2003)
3. Dai, H.: On the symmetric solutions of linear matrix equations. Linear Algebra Appl. **131**, 1–7 (1990)
4. Chu, K.E.: Symmetric solutions of linear matrix equations by matrix decompositions. Linear Algebra Appl. **119**, 35–50 (1989)
5. Mitra, S.K.: Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$. Proc. Camb. Philos. Soc. **74**, 213–216 (1973)
6. Peng, X.Y., Hu, X.Y., Zhang, L.: The reflexive and anti-reflexive solutions of the matrix equation $A^*XB = C$. J. Comput. Appl. Math. **200**, 749–760 (2007)
7. Peng, Y.X., Hu, X.Y., Zhang, L.: An iteration method for the symmetric solutions and the optimal approximation solution of the matrix equation $AXB = C$. Appl. Math. Comput. **160**, 763–777 (2005)
8. Peng, Y.X., Hu, X.Y., Zhang, L.: An iteration method for symmetric solutions and optimal approximation solution of the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$. Appl. Math. Comput. **183**, 1127–1137 (2006)
9. Peng, Z.Y.: An iterative method for the least-squares symmetric solution of the linear matrix equation $AXB = C$. Appl. Math. Comput. **170**, 711–723 (2005)
10. Xie, D.X., Hu, X.Y., Sheng, Y.P.: The solvability conditions for the inverse eigenproblems of symmetric and generalized centro-symmetric matrices and their approximations. Linear Algebra Appl. **418**, 142–152 (2006)