Proceedings of the Royal Society of Edinburgh, **139A**, 367–379, *2009*

Two non-trivial solutions for a non-homogeneous Neumann problem: an Orlicz–Sobolev space setting

Alexandru Kristály
Department of Economics, University of Babeş-Bolyai,
400591 Cluj-Napoca, Romania and
Department of Mathematics, Central European University,
1051 Budapest, Hungary (alexandrukristaly@yahoo.com)

Mihai Mihăilescu
Department of Mathematics, Central European University,
1051 Budapest, Hungary and
Department of Mathematics, University of Craiova,
200585 Craiova, Romania (mmihailes@yahoo.com)

Vicenţiu Rădulescu
Department of Mathematics, University of Craiova,
200585 Craiova, Romania and
Institute of Mathematics 'Simion Stoilow' of the Romanian Academy,
014700 Bucharest, Romania (vicentiu.radulescu@math.cnrs.fr)

(MS received 21 February 2007; accepted 27 March 2008)

In this paper we study a non-homogeneous Neumann-type problem which involves a nonlinearity satisfying a non-standard growth condition. By using a recent variational principle of Ricceri, we establish the existence of at least two non-trivial solutions in an appropriate Orlicz–Sobolev space.

1. Introduction and the main result

In this paper we consider the problem

$$\left.\begin{aligned} -\operatorname{div}(a(|\nabla u(x)|)\nabla u(x)) + a(|u(x)|)u(x) &= \lambda f(x, u(x)) &&\text{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) &= 0 &&\text{for } x \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and λ is a positive parameter. Throughout this paper we assume that the function $a : (0, \infty) \to \mathbb{R}$ is such that the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases} \tag{1.2}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Equation (1.1) has been widely studied in the homogeneous case when $a(t) = t^{p-2}$, $p > 1$, which corresponds to a problem involving the classical p-Laplacian (see [4,6,11,31]). The purpose of this paper is to consider (1.1) in the aforementioned general framework, when the nonlinear term f satisfies a non-standard growth condition at infinity. To be more precise, we first introduce the functions

$$\Phi(t) = \int_0^t \phi(s)\,\mathrm{d}s, \Phi^\star(t) = \int_0^t \phi^{-1}(s)\,\mathrm{d}s \quad \text{for all } t \in \mathbb{R}. \tag{1.3}$$

We observe that Φ is a *Young function*, that is, $\Phi(0) = 0$, Φ is convex and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty,$$

then Φ is called an *N-function*. The function $\Phi^\star$ is called the *complementary* function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s); s \geqslant 0\} \quad \text{for all } t \geqslant 0.$$

We observe that $\Phi^\star$ is also an N-function and the following Young inequality holds:

$$st \leqslant \Phi(s) + \Phi^\star(t) \quad \text{for all } s, t \geqslant 0.$$

Throughout this paper we assume that

$$1 < \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)} \leqslant \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} < \infty. \tag{Φ_0}$$

Due to assumption (Φ_0), we may define the numbers

$$p_0 := \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \quad \text{and} \quad p^0 := \sup_{t>0} \frac{t\phi(t)}{\Phi(t)}.$$

Note that for $a(t) = t^{p-2}$, $p > 1$, one has $p_0 = p^0 = p$.

On the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ we will assume that

(f0) there exist $c_0 > 0$ and $0 < s < p_0 - 1$ such that $|f(x,t)| \leqslant c_0(1 + |t|^s)$ for every $(x,t) \in \Omega \times \mathbb{R}$,

(f1) there exists $b \in \mathbb{R}$ such that

$$B_F = \int_\Omega F(x,b)\,\mathrm{d}x > 0,$$

where $F(x,t) = \int_0^t f(x,w)\,\mathrm{d}w$, $t \in \mathbb{R}$,

(f2) there exists $\delta > 0$ such that $f(x,t)t \leqslant 0$ for every $x \in \Omega$ and $t \in [-\delta, \delta]$.

Roughly speaking, the growth of $f(x,\cdot)$ is (p_0-1)-*sublinear at infinity* (see (f0)). In this setting, the presence of the eigenvalue $\lambda > 0$ in (1.1) is indispensable. Indeed, if we analyse even the simplest case $a(t) = 1$ that corresponds to the Laplace equation and we assume that $f(x,\cdot)$ is uniformly Lipschitz with Lipschitz constant $L > 0$ (uniformly for $x \in \Omega$), then (1.1) has only the *trivial* weak solution whenever $\lambda < L^{-1}$. Moreover, (f2) implies in particular that $f(x,0) = 0$ for every $x \in \Omega$; thus, $u = 0$ can always be considered a solution of problem (1.1). However, assuming finally that

$$N < p_0 < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}, \tag{Φ_1}$$

we may prove the following multiplicity result.

THEOREM 1.1. *Assume that* (Φ_0) *and* (Φ_1) *hold and that the function* $[0,\infty) \ni t \to \Phi(\sqrt{t})$ *is convex. Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function which satisfies (f0)–(f2).*

Then there exist a non-empty open interval $\Lambda \subset (0, 2\Phi(b)|\Omega|B_F^{-1})$ *and* $\mu > 0$ *such that for any* $\lambda \in \Lambda$ *problem (1.1) has at least two non-trivial weak solutions whose norms are less than* μ.

The precise notion of weak solutions for (1.1) will be given in § 2. This step will be possible by introducing an Orlicz–Sobolev space setting, due to the fact that the operator in the divergence form is non-homogeneous. In particular, in the homogeneous (p-Laplace operator) case, theorem 1.1 extends known results (see, for instance, [4, 6, 31]); moreover, we give an estimate for the interval $\Lambda \subset (0,\infty)$ where problem (1.1) has at least two non-trivial weak solutions.

On the other hand, we point out that it is possible for the technical assumption, i.e. the function $[0,\infty) \ni t \to \Phi(\sqrt{t})$ is convex, not to be a necessary condition. Actually, it will be used in the proof of theorem 1.1 in order to obtain a Clarkson-type inequality for the function Φ, i.e.

$$\begin{aligned}\frac{1}{2}\biggl[\int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\biggr] \\ \geqslant \int_\Omega \Phi\biggl(\biggl|\frac{\nabla u + \nabla v}{2}\biggr|\biggr)\,\mathrm{d}x + \int_\Omega \Phi\biggl(\biggl|\frac{\nabla u - \nabla v}{2}\biggr|\biggr)\,\mathrm{d}x\end{aligned} \tag{1.4}$$

for any $u, v \in W^1L_\Phi(\Omega)$, where $W^1L_\Phi(\Omega)$ is an Orlicz–Sobolev functional space that will be defined in the next section. Obviously, inequality (1.4) extends the *classical* Clarkson inequality, obtained for the homogeneous function $\Phi(t) = t^p$ with $p \geqslant 2$ (see [21] for more details). Unfortunately, at this stage we cannot say firmly whether an inequality of type (1.4) can be stated for a class of functions which do not satisfy the fact that $t \to \Phi(\sqrt{t})$ is convex. Since, for the moment, the above quoted condition is the only one that we have found in the literature to yield to inequalities of type (1.4), we have inserted it in the hypotheses of theorem 1.1 instead of the assumption that the function Φ satisfies inequality (1.4). The necessity of the condition remains an open question.

The first general existence result using the theory of monotone operators in Orlicz–Sobolev spaces was obtained by Donaldson [9] and Gossez [13, 14]. Other recent works that put the problem into this framework include [7,8,12,15,22,25–27]. In these papers, the existence results are obtained by means of variational techniques, monotone operator methods or fixed-point and degree theory arguments. Concerning the boundary-value problems with Neumann boundary condition, we point out the existence and multiplicity results obtained by Halidias and Le [16].

In the next section we recall some basic facts on Orlicz–Sobolev spaces; we will prove theorem 1.1 in the last section.

2. Orlicz–Sobolev setting

Let $\phi : \mathbb{R} \to \mathbb{R}$ and Φ, Φ^* be as in (1.2) and (1.3), respectively. The Orlicz space $L_\Phi(\Omega)$ defined by the N-function Φ [1, 2, 7] is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{ \int_\Omega uv \,\mathrm{d}x;\ \int_\Omega \Phi^\star(|v|)\,\mathrm{d}x \leqslant 1 \right\} < \infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{ k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right) \mathrm{d}x \leqslant 1 \right\}.$$

For Orlicz spaces, Hölder's inequality reads as follows (see [29, inequality (4), p. 79]):

$$\int_\Omega uv \,\mathrm{d}x \leqslant 2\|u\|_{L_\Phi}\|v\|_{L_{\Phi^\star}} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^\star}(\Omega).$$

We denote by $W^1 L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space for problem (1.1), defined by

$$W^1 L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \dots, N \right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi$$

(see [2, 7, 13]). The spaces $L_\Phi(\Omega)$ and $W^1 L_\Phi(\Omega)$ are studied in depth in [1, 2, 19, 23, 29]. These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p$ is assumed by a more general convex function $\Phi(t)$. One of the main features of Orlicz–Sobolev spaces is that they fill a gap in the classical theory of Sobolev embeddings. Indeed, if $kp = N$ and $p > 1$, then $W^{k,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for any $p \leqslant q < \infty$, but there is no *smallest* target L^q space for these embeddings, in the sense that $W^{k,p}(\Omega) \nsubseteq L^\infty(\Omega)$. However, if the class of target spaces is enlarged to contain Orlicz spaces, then, as shown in [32] (see also [17]), the best such target space is $L_\Phi(\Omega)$, where $\Phi(t) = \exp(|t|^{p/(p-1)}) - 1$. This inequality has been extended to Lorentz spaces by Malý and Pick [24]. We also point out that many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces by Donaldson and Trudinger [10].

We say that $u \in W^1L_\Phi(\Omega)$ is a *weak solution* for problem (1.1) if

$$\int_\Omega a(|\nabla u|)\nabla u \nabla v \,\mathrm{d}x + \int_\Omega a(|u|)uv\,\mathrm{d}x - \lambda \int_\Omega f(x,u)v\,\mathrm{d}x = 0 \quad \text{for all } v \in W^1L_\Phi(\Omega).$$

Hypothesis (Φ_0) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2-condition (at infinity) (see [2, p. 232] and [7]). In particular, both (Φ, Ω) and (Φ^*, Ω) are Δ-regular (see [2, p. 232]). Consequently, the spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces (see [2, pp. 241, 247]).

Remark 2.1. Using lemma D.2 of [7] it follows that $W^1L_\Phi(\Omega)$ is continuously embedded in $W^{1,p_0}(\Omega)$. On the other hand, since we assume that $p_0 > N$, we deduce that $W^{1,p_0}(\Omega)$ is compactly embedded in $C(\bar\Omega)$. Thus, we deduce that $W^1L_\Phi(\Omega)$ is compactly embedded in $C(\bar\Omega)$. Defining $\|u\|_\infty = \sup_{x\in\bar\Omega}|u(x)|$, we find a positive constant $c > 0$ such that

$$\|u\|_\infty \leqslant c\|u\|_{1,\Phi} \quad \text{for all } u \in W^1L_\Phi(\Omega).$$

We point out certain useful properties regarding the norms on Orlicz–Sobolev spaces.

Lemma 2.2. *On $W^1L_\Phi(\Omega)$ the norms*

$$\begin{aligned}
\|u\|_{1,\Phi} &= \||\nabla u|\|_\Phi + \|u\|_\Phi,\\
\|u\|_{2,\Phi} &= \max\{\||\nabla u|\|_\Phi, \|u\|_\Phi\},\\
\|u\| &= \inf\left\{\mu > 0; \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right)\right]\mathrm{d}x \leqslant 1\right\}
\end{aligned}$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\| \leqslant 2\|u\|_{2,\Phi} \leqslant 2\|u\|_{1,\Phi} \leqslant 4\|u\|.$$

Proof. First, we point out that $\|\cdot\|_{1,\Phi}$ and $\|\cdot\|_{2,\Phi}$ are equivalent, since

$$\|u\|_{2,\Phi} \leqslant \|u\|_{1,\Phi} \leqslant 2\|u\|_{2,\Phi} \quad \text{for all } u \in W^1L_\Phi(\Omega). \tag{2.1}$$

In the following, we assume that $u \neq 0$. We remark that

$$\int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|_\Phi}\right)\mathrm{d}x \leqslant 1, \qquad \int_\Omega \Phi\left(\frac{|\nabla u(x)|}{\||\nabla u|\|_\Phi}\right)\mathrm{d}x \leqslant 1 \tag{2.2}$$

and

$$\int_\Omega \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right)\right]\mathrm{d}x \leqslant 1. \tag{2.3}$$

By (2.3) we obtain

$$\int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|}\right)\mathrm{d}x \leqslant 1 \quad \text{and} \quad \int_\Omega \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right)\mathrm{d}x \leqslant 1.$$

Taking into account the way in which $\|\cdot\|_\Phi$ is defined, we find

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi \leqslant 2\|u\| \quad \text{for all } u \in W^1L_\Phi(\Omega). \tag{2.4}$$

On the other hand, since

$$\Phi(t) \leqslant \frac{t\phi(t)}{p_0} \quad \text{for all } t > 0,$$

with $p_0 > N$, by [8, lemma C.4(ii)] we deduce in particular that

$$\Phi(2t) \geqslant 2\Phi(t) \quad \text{for all } t > 0.$$

Thus, we deduce that, for all $u \in W^1L_\Phi(\Omega)$, $x \in \Omega$,

$$2\Phi\left(\frac{|u(x)|}{2\|u\|_{2,\Phi}}\right) \leqslant \Phi\left(\frac{|u(x)|}{\|u\|_{2,\Phi}}\right) \quad \text{and} \quad 2\Phi\left(\frac{|\nabla u(x)|}{2\|u\|_{2,\Phi}}\right) \leqslant \Phi\left(\frac{|\nabla u(x)|}{\|u\|_{2,\Phi}}\right).$$

It follows that

$$\int_\Omega \left[\Phi\left(\frac{|u(x)|}{2\|u\|_{2,\Phi}}\right) + \Phi\left(\frac{|\nabla u(x)|}{2\|u\|_{2,\Phi}}\right)\right] \mathrm{d}x \leqslant \frac{1}{2}\left\{\int_\Omega \left[\Phi\left(\frac{|u(x)|}{\|u\|_{2,\Phi}}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|_{2,\Phi}}\right)\right] \mathrm{d}x\right\}. \tag{2.5}$$

But, since

$$\|u\|_{2,\Phi} \geqslant \|u\|_\Phi \quad \text{and} \quad \|u\|_{2,\Phi} \geqslant \||\nabla u|\|_\Phi \quad \text{for all } u \in W^1L_\Phi(\Omega),$$

we get

$$\frac{|u(x)|}{\|u\|_\Phi} \geqslant \frac{|u(x)|}{\|u\|_{2,\Phi}} \quad \text{and} \quad \frac{|\nabla u(x)|}{\||\nabla u|\|_\Phi} \geqslant \frac{|\nabla u(x)|}{\|u\|_{2,\Phi}} \quad \text{for all } u \in W^1L_\Phi(\Omega), x \in \Omega. \tag{2.6}$$

Taking into account the fact that Φ is increasing on $[0, \infty)$, by (2.5), (2.6) and (2.2), we get

$$\begin{aligned}\int_\Omega \left[\Phi\left(\frac{|u(x)|}{2\|u\|_{2,\Phi}}\right) + \Phi\left(\frac{|\nabla u(x)|}{2\|u\|_{2,\Phi}}\right)\right] \mathrm{d}x &\leqslant \frac{1}{2}\left\{\int_\Omega \left[\Phi\left(\frac{|u(x)|}{\|u\|_\Phi}\right) + \Phi\left(\frac{|\nabla u(x)|}{\||\nabla u|\|_\Phi}\right)\right] \mathrm{d}x\right\} \\ &\leqslant 1,\end{aligned}$$

for all $u \in W^1L_\Phi(\Omega)$. Thus, we conclude that

$$\|u\| \leqslant 2\|u\|_{2,\Phi} \quad \text{for all } u \in W^1L_\Phi(\Omega). \tag{2.7}$$

By relations (2.1), (2.4) and (2.7) we deduce that lemma 2.2 holds. □

Lemma 2.3. *The following relations hold:*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] \,\mathrm{d}x \geqslant \|u\|^{p_0} \quad \textit{for all } u \in W^1L_\Phi(\Omega) \textit{ with } \|u\| > 1,$$

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] \,\mathrm{d}x \geqslant \|u\|^{p^0} \quad \textit{for all } u \in W^1L_\Phi(\Omega) \textit{ with } \|u\| < 1.$$

Proof. First, assume that $\|u\| > 1$. Let $\beta \in (1, \|u\|)$. By [8, lemma C.4(ii)] we have

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] \,\mathrm{d}x \geqslant \beta^{p_0} \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\beta}\right) + \Phi\left(\frac{|\nabla u(x)|}{\beta}\right)\right] \mathrm{d}x \geqslant \beta^{p_0}.$$

Letting $\beta \nearrow \|u\|$, we find

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \geqslant \|u\|^{p_0} \quad \text{for all } u \in W^1 L_\Phi(\Omega) \text{ with } \|u\| > 1.$$

Next, assume that $\|u\| < 1$. Let $\xi \in (0, \|u\|)$. By the definition of p^0, it is easy to prove that

$$\Phi(t) \geqslant \tau^{p^0}\Phi(t/\tau) \quad \text{for all } t > 0,\ \tau \in (0,1).$$

Using the above relation we have

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \geqslant \xi^{p^0} \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\xi}\right) + \Phi\left(\frac{|\nabla u(x)|}{\xi}\right)\right]\mathrm{d}x. \quad (2.8)$$

Defining $v(x) = u(x)/\xi$ for all $x \in \Omega$, we have $\|v\| = \|u\|/\xi > 1$. Using the first inequality of this lemma we find

$$\int_\Omega [\Phi(|v(x)|) + \Phi(|\nabla v(x)|)]\,\mathrm{d}x \geqslant \|v\|^{p_0} > 1. \quad (2.9)$$

Relations (2.8) and (2.9) show that

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \geqslant \xi^{p^0}.$$

Letting $\xi \nearrow \|u\|$ in the above inequality, we obtain

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \geqslant \|u\|^{p^0} \quad \text{for all } u \in W^1 L_\Phi(\Omega) \text{ with } \|u\| < 1.$$

The proof of lemma 2.3 is complete. □

3. Proof of theorem 1.1

The key argument in the proof of our main result is a *three-critical-point* theorem due to Ricceri [30]. This result is widely applied to solve various elliptic problems; we refer the reader to [4–6, 20, 31]. Ricceri's result goes back to an elementary property established by Pucci and Serrin (see [30, theorem 3]) which asserts that if a functional of class C^1 defined on a real Banach space has two local minima, then it has a third critical point. This is an auxiliary result related to a problem of Rabinowitz [28], who raised the question whether critical points of mountain-pass type must necessarily be saddle points. To the best of our knowledge, the first *three-critical-point* property was found by Krasnoselskii [18]; he showed that if f is a coercive C^1 functional defined on a finite-dimensional space having a non-degenerate critical point x_0 (that is, the *topological index* $\operatorname{ind} f'(x_0)(0)$ is non-zero) which is not a global minimum, then f admits a third critical point. This result was extended to infinite-dimensional Banach spaces by Amann [3].

We recall in what follows a sharper version of Ricceri's theorem, which is due to Bonanno (see [5, theorem 2.1]).

THEOREM 3.1. *Let E be a separable and reflexive real Banach space and let $J, I : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $u_0 \in E$ such that $J(u_0) = I(u_0) = 0$ and $J(u) \geqslant 0$ for every $u \in E$ and that there exists $u_1 \in E$, $r > 0$ such that*

(i) $r < J(u_1)$,

(ii) $\sup_{J(u)<r} I(u) < r(I(u_1)/J(u_1))$.

Furthermore, set

$$\bar{a} = \frac{\zeta r}{r(I(u_1)/J(u_1)) - \sup_{J(u)<r} I(u)},$$

with $\zeta > 1$, and assume that the functional $J - \lambda I$ is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and

(iii) $\lim_{\|u\|\to+\infty}(J(u) - \lambda I(u)) = +\infty$ *for every* $\lambda \in [0, \bar{a}]$.

Then, there exist a non-empty open interval $\Lambda \subset [0, \bar{a}]$ and a number $\mu > 0$ such that, for each $\lambda \in \Lambda$, the equation $J'(u) - \lambda I'(u) = 0$ admits at least three solutions in E having the norm less than μ.

From now on, we assume that the hypotheses of theorem 1.1 are satisfied. Let $E = W^1 L_\Phi(\Omega)$ be the Orlicz–Sobolev space from § 2. We further define the functionals $J, I : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega (\Phi(|\nabla u|) + \Phi(|u|))\,\mathrm{d}x \quad \text{and} \quad I(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x.$$

Similar arguments as those used in [12, lemma 3.4] and [7, lemma 2.1] imply that $J, I \in C^1(E, \mathbb{R})$ with the derivatives given by

$$\langle J'(u), v\rangle = \int_\Omega a(|\nabla u|)\nabla u \nabla v\,\mathrm{d}x + \int_\Omega a(|u|)uv\,\mathrm{d}x,$$
$$\langle I'(u), v\rangle = \int_\Omega f(x, u)v\,\mathrm{d}x$$

for any u, $v \in E$. Let us observe that $u \in E$ is a weak solution of equation (1.1) if there exists $\lambda > 0$ such that u is a critical point of the functional $J - \lambda I$. Therefore, we can seek for weak solutions of problem (1.1) by applying theorem 3.1. In the following, we will verify all the hypotheses of theorem 3.1. In order to do this, we first prove the following lemma.

LEMMA 3.2. *$J' : E \to E^\star$ has a continuous inverse operator on $E^\star$.*

Proof. We will use [33, theorem 26.A(d)]; namely, it is sufficient to verify that J' is coercive, hemicntinuous and uniformly monotone.

Indeed, since Φ is convex it follows that J is also convex. Thus, we have

$$J(u) \leqslant \langle J'(u), u\rangle \quad \text{for all } u \in E.$$

By lemma 2.3 it is clear that for any $u \in E$ with $\|u\| > 1$ we have

$$\frac{\langle J'(u), u\rangle}{\|u\|} \geqslant \frac{J(u)}{\|u\|} \geqslant \|u\|^{p_0-1}.$$

Thus,

$$\lim_{\|u\|\to\infty} \frac{\langle J'(u), u\rangle}{\|u\|} = \infty,$$

i.e. J' is coercive.

The fact that J' is hemicontinuous can be verified using standard arguments.

Finally, we show that J' is uniformly monotone. Indeed, since Φ is convex, we have

$$\Phi(|\nabla u(x)|) \leqslant \Phi\left(\left|\frac{\nabla u(x) + \nabla v(x)}{2}\right|\right) + a(|\nabla u(x)|)\nabla u(x) \cdot \frac{\nabla u(x) - \nabla v(x)}{2}$$

and

$$\Phi(|\nabla v(x)|) \leqslant \Phi\left(\left|\frac{\nabla u(x) + \nabla v(x)}{2}\right|\right) + a(|\nabla v(x)|)\nabla v(x) \cdot \frac{\nabla v(x) - \nabla u(x)}{2}$$

for every $u, v \in E$ and $x \in \Omega$. Adding the above two relations and integrating over Ω we find

$$\begin{aligned}\frac{1}{2}\int_\Omega &(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v)\,\mathrm{d}x \\ &\geqslant \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x - 2\int_\Omega \Phi\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right)\mathrm{d}x \qquad (3.1)\end{aligned}$$

for any u, $v \in E$.

On the other hand, since $\Phi : [0, \infty) \to \mathbb{R}$ is an increasing, continuous function with $\Phi(0) = 0$, and $t \mapsto \Phi(\sqrt{t})$ is convex, we deduce by [21] that

$$\begin{aligned}\frac{1}{2}\bigg[\int_\Omega &\Phi(|\nabla u|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\bigg] \\ &\geqslant \int_\Omega \Phi\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right)\mathrm{d}x + \int_\Omega \Phi\left(\left|\frac{\nabla u - \nabla v}{2}\right|\right)\mathrm{d}x \qquad (3.2)\end{aligned}$$

for any u, $v \in E$.

By (3.1) and (3.2) it follows that

$$\begin{aligned}\int_\Omega &(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v)\,\mathrm{d}x \\ &\geqslant 4\int_\Omega \Phi\left(\left|\frac{\nabla u - \nabla v}{2}\right|\right)\mathrm{d}x \quad \text{for all } u, v \in E. \qquad (3.3)\end{aligned}$$

Similarly,

$$\int_\Omega (a(|u|)u - a(|v|)v)(u - v)\,\mathrm{d}x \geqslant 4\int_\Omega \Phi\left(\left|\frac{u - v}{2}\right|\right)\mathrm{d}x \quad \text{for all } u, v \in E. \qquad (3.4)$$

Relations (3.3) and (3.4) yield

$$\langle J'(u) - J'(v), u - v \rangle \geqslant 4J\left(\frac{u-v}{2}\right).$$

Define the function $\alpha : [0, \infty) \to [0, \infty)$ by

$$\alpha(t) = \frac{1}{2^{p^0-2}} \begin{cases} t^{p^0-1} & \text{for } t \leqslant 1, \\ t^{p_0-1} & \text{for } t \geqslant 1. \end{cases}$$

It is easy to check that α is an increasing function with $\alpha(0) = 0$ and $\lim_{t\to\infty} \alpha(t) = \infty$. Taking into account the above information and lemma 2.3, we deduce that

$$\langle J'(u) - J'(v), u - v \rangle \geqslant \alpha(\|u - v\|)\|u - v\| \quad \text{for all } u, v \in E,$$

i.e. J' is uniformly monotone, which concludes our proof. □

Now, we will verify the hypotheses of theorem 3.1 in three steps.

STEP 1. For every $\lambda > 0$, the functional $J - \lambda I$ is coercive, i.e. (iii) is verified.

Indeed, by lemma 2.3 we deduce that for any $u \in E$ with $\|u\| > 1$ we have $J(u) \geqslant \|u\|^{p_0}$. On the other hand, by (f0), there exists $c_1 > 0$ such that

$$\int_\Omega F(x, u(x))\,\mathrm{d}x \leqslant c_1 \int_\Omega (|u| + |u|^{s+1})\,\mathrm{d}x \leqslant c_1|\Omega|(\|u\|_\infty + \|u\|_\infty^{s+1}) \quad \text{for all } u \in E.$$

Since E is compactly embedded into $C(\bar{\Omega})$ (see remark 2.1), due to lemma 2.2, it follows that there exists $c_2 > 0$ such that

$$J(u) - \lambda I(u) \geqslant \|u\|^{p_0} - \lambda c_2|\Omega|(\|u\| + \|u\|^{s+1}) \quad \text{for all } u \in E.$$

Since $1 < s + 1 < p_0$, it follows that

$$\lim_{\|u\|\to\infty} (J(u) - \lambda I(u)) = \infty \quad \text{for all } \lambda > 0;$$

thus (iii) is verified.

STEP 2. For every $\lambda > 0$, the functional $J - \lambda I$ is sequentially weakly lower semicontinuous and satisfies the Palais–Smale condition.

The fact that E is compactly embedded into $C(\bar{\Omega})$ implies that the operator $I' : E \to E^\star$ is compact. Consequently, the functional $I : X \to \mathbb{R}$ is sequentially weakly continuous (see [34, corollary 41.9]). On the other hand, the convexity of $J : X \to \mathbb{R}$ implies the sequentially weak lower semicontinuity of J. This proves the first part.

Combining step 1, lemma 3.2 and the fact that $I' : E \to E^\star$ is compact, we obtain that $J - \lambda I$ satisfies the Palais–Smale condition (see [34, example 38.25]).

STEP 3. Let $0 < r < \min\{1, (\delta/2c)^{p^0}, \Phi(b)|\Omega|\}$ and $u_1(x) = b \in E$. Then (i) and (ii) are verified.

First, we observe that $b \neq 0$ (which appears in (f1)). Therefore, $\Phi(b) = \Phi(-b) > 0$, i.e. one may choose $r > 0$ as above. Now, we have

$$J(u_1) = \int_\Omega (\Phi(|\nabla u_1|) + \Phi(|u_1|))\,\mathrm{d}x = \int_\Omega \Phi(|b|) = \Phi(b)|\Omega| > r,$$

i.e. (i) is verified.

Now, let $J(u) < r$. Then, by lemma 2.3 (and $r < 1$), we have $\|u\|^{p^0} \leqslant J(u) < r$. Therefore, $\|u\| \leqslant \delta/2c$. By remark 2.1 and lemma 2.2, we have

$$|u(x)| \leqslant \|u\|_\infty \leqslant c\|u\|_{1,\Phi} \leqslant 2c\|u\| \leqslant \delta \quad \text{for all } x \in \Omega. \tag{3.5}$$

On the other hand, by (f2), we have that

$$F(x,t) = F(x,t) - F(x,0) = f(x,\theta t)t = \frac{1}{\theta} f(x,\theta t)\theta t \leqslant 0 \quad (\text{with } \theta \in (0,1))$$

for every $x \in \Omega$ and $t \in [-\delta, \delta]$. Consequently, for every $u \in E$, complying with $J(u) < r$, we have

$$I(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x \leqslant 0$$

(see (3.5)); thus,

$$\sup_{J(u)<r} I(u) \leqslant 0. \tag{3.6}$$

But, by (f1), we have

$$r\frac{I(u_1)}{J(u_1)} = \frac{rB_F}{\Phi(b)|\Omega|} > 0,$$

which proves (ii).

Proof of theorem 1.1. It is clear that $I(0) = J(0) = 0$ and $J(u) \geqslant 0$ for every $u \in E$. Choosing $u_0 = 0$ and taking into account steps 1–3, all the hypotheses of theorem 3.1 are verified. Setting

$$\bar{a} = \frac{2r}{r(I(u_1)/J(u_1)) - \sup_{J(u)<r} I(u)},$$

there exist a non-empty open interval $\Lambda \subset [0, \bar{a}]$ and a number $\mu > 0$ such that, for each $\lambda \in \Lambda$, the equation $J'(u) - \lambda I'(u) = 0$ admits at least three solutions in E (thus, at least two non-trivial weak solutions for (1.1)) having the norm less than μ. Moreover, due to (3.6), we have

$$\bar{a} \leqslant \frac{2r}{r(I(u_1)/J(u_1))} = \frac{2J(u_1)}{I(u_1)} = \frac{2\Phi(b)|\Omega|}{B_F},$$

which completes the proof of theorem 1.1. □

Example 3.3. Let us consider the problem

$$\left.\begin{aligned} -\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{\log(1+|\nabla u|)}\right) + \frac{|u|^{p-2}u}{\log(1+|u|)} &= \lambda \ln(1 + (u-1)u_+^{q(x)}) && \text{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{for } x \in \partial\Omega, \end{aligned}\right\} \tag{3.7}$$

where p is a real number such that $p > N+1$ and $q \in C(\bar{\Omega})$ satisfies $2 < q(x) < p-1$ for any $x \in \bar{\Omega}$ and $u_+ = \max(u, 0)$.

We define

$$\phi(t) = \frac{|t|^{p-2}}{\log(1+|t|)} t \quad \text{for } t \neq 0 \text{ and } \phi(0) = 0$$

and

$$\Phi(t) = \int_0^t \phi(s)\, \mathrm{d}s.$$

An easy computation shows that the function $[0, \infty) \ni t \mapsto \Phi(\sqrt{t})$ is convex. Moreover, by [8, example 3, p. 243] we have

$$p_0 = p - 1 < p^0 = p = \liminf_{t \to \infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions (Φ_0) and (Φ_1) are verified.

Now we define the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,t) = \ln(1 + (t-1)t_+^{q(x)}) \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}.$$

Then $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} F(x,t) = t \ln(1 + (t-1)t_+^{q(x)}) - (q(x)+1)t_+ \\ + \int_0^{t_+} \frac{q(x) + 1 - s^{q(x)}}{1 + s^{q(x)+1} - s^{q(x)}}\, \mathrm{d}s \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}. \end{aligned}$$

Clearly, f is a Carathéodory function and (f0) is satisfied by choosing $s = 1$. Moreover, for sufficiently large $b > 0$, (f1) is also verified. Finally, (f2) is verified for $\delta = 1$. Consequently, we can apply theorem 1.1, and hence problem (3.7) has at least two non-trivial solutions for certain eigenvalues $\lambda > 0$.

Acknowledgments

A.K. is supported by the CNCSIS Project no. AT 8/70 and by Grant PNII ID no. 527/2007. V.R. is supported by Grant no. 2-CEx06-11-18/2006. M.M. and V.R. are also supported by Grant CNCSIS PNII ID no. 79/2007.

References

1 D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*. Grundlehren der Mathematischen Wissenschaften, vol. 314 (Springer, 1996).

2 R. A. Adams. *Sobolev spaces* (Academic, 1975).

3 H. Amann. A note on degree theory for gradient mappings. *Proc. Am. Math. Soc.* **85** (1982), 591–597.

4 G. Anello and G. Cordaro. Existence of solutions of the Neumann problem involving p-Laplacian via variational principle of Ricceri. *Arch. Math.* **79** (2002), 274–287.

5 G. Bonanno. Some remarks on a three critical points theorem. *Nonlin. Analysis* **54** (2003), 651–665.

6 G. Bonanno and P. Candito. Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian. *Arch. Math.* **80** (2003), 424–429.

7 Ph. Clément, M. García-Huidobro, R. Manásevich and K. Schmitt. Mountain pass type solutions for quasilinear elliptic equations. *Calc. Var. PDEs* **11** (2000), 33–62.

8 Ph. Clément, B. de Pagter, G. Sweers and F. de Thélin. Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces. *Mediterr. J. Math.* **1** (2004), 241–267.

9 T. K. Donaldson. Nonlinear elliptic boundary value problems in Orlicz–Sobolev spaces. *J. Diff. Eqns* **10** (1971), 507–528.

10 T. K. Donaldson and N. S. Trudinger. Orlicz–Sobolev spaces and imbedding theorems. *J. Funct. Analysis* **8** (1971), 52–75.

11 F. Faraci and A. Kristály. On an open question of Ricceri concerning a Neumann problem. *Glasgow Math. J.* **49** (2007), 189–195.

12 M. Garciá-Huidobro, V. K. Le, R. Manásevich and K. Schmitt. On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev space setting. *Nonlin. Diff. Eqns Applic.* **6** (1999), 207–225.

13 J. P. Gossez. Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. *Trans. Am. Math. Soc.* **190** (1974), 163–205.

14 J. P. Gossez. *A strongly nonlinear elliptic problem in Orlicz–Sobolev spaces*, Proceedings of Symposia in Pure Mathematics, vol. 45, pp. 455–462 (Providence, RI: American Mathematical Society, 1986).

15 J. P. Gossez and R. Manásevich. On a nonlinear eigenvalue problem in Orlicz–Sobolev spaces. *Proc. R. Soc. Edinb.* A **132** (2002), 891–909.

16 N. Halidias and V. K. Le. Multiple solutions for quasilinear elliptic Neumann problems in Orlicz–Sobolev spaces. *Boundary Value Problems* **3** (2005), 299–306.

17 J. A. Hempel, G. R. Morris and N. S. Trudinger. On the sharpness of a limiting case of the Sobolev imbedding theorem. *Bull. Austral. Math. Soc.* **3** (1970), 369–373.

18 M. A. Krasnoselskii. *The operator of translation along the trajectories of differential equations* (Moscow: Nauka, 1963). (In Russian.)

19 M. A. Krasnoselskii and Ya. B. Rutiskii. *Convex functions and Orlicz spaces* (Groningen: Noordhoff, 1961).

20 A. Kristály. Existence of two non-trivial solutions for a class of quasilinear elliptic variational systems on strip-like domains. *Proc. Edinb. Math. Soc.* **48** (2005), 465–477.

21 J. Lamperti. On the isometries of certain function-spaces. *Pac. J. Math.* **8** (1958), 459–466.

22 V. K. Le and K. Schmitt. Quasilinear elliptic equations and inequalities with rapidly growing coefficients. *J. Lond. Math. Soc.* **62** (2000), 852–872.

23 W. Luxemburg. Banach function spaces. PhD thesis, Technische Hogeschool te Delft, The Netherlands (1955).

24 J. Malý and L. Pick. An elementary proof of sharp Sobolev embeddings. *Proc. Am. Math. Soc.* **130** (2002), 555–563.

25 M. Mihăilescu and V. Rădulescu. A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. *Proc. R. Soc. Lond.* A **462** (2006), 2625–2641.

26 M. Mihăilescu and V. Rădulescu. Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz–Sobolev space setting. *J. Math. Analysis Applic.* **330** (2007), 416–432.

27 M. Mihăilescu and V. Rădulescu. On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. *Proc. Am. Math. Soc.* **135** (2007), 2929–2937.

28 P. H. Rabinowitz. Variational methods for nonlinear eigenvaalue problems. In *Eigenvalues of nonlinear problems*, pp. 139–195 (Rome: Edizioni Cremonese, 1974).

29 M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces* (New York: Marcel Dekker, 1991).

30 B. Ricceri. On a three critical points theorem. *Arch. Math.* **75** (2000), 220–226.

31 B. Ricceri. Three solutions for a Neumann problem. *Topolog. Meth. Nonlin. Analysis* **20** (2002), 275–281.

32 N. Trudinger. On embeddings into Orlicz spaces and some applications. *J. Math. Mech.* **17** (1967), 473–483.

33 E. Zeidler. *Nonlinear functional analysis and its applications*, vol. II/B (Springer, 1985).

34 E. Zeidler. *Nonlinear functional analysis and its applications*, vol. III (Springer, 1985).

(*Issued 10 April 2009*)