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Dynamic response of a torsional micromirror to electrostatic force and mechanical shock

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A R T I C L E I N F O

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Dedicated to the memory of Professor Ioannis Vardoulakis.

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A B S T R A C T

In this paper a new modified couple stress model is developed for the Saint–Venant torsion problem of micro-bars of arbitrary cross-section. The proposed model is derived from a modified couple stress theory and has only one material length scale parameter. Using a variational procedure the governing differential equation and the associated boundary conditions are derived in terms of the warping function. This is a fourth order partial differential equation representing the analog of a Kirchhoff plate having the shape of the cross-section and subjected to a uniform tensile membrane force with mixed Neumann boundary conditions. Since the fundamental solution of the equation is known, the problem could be solved using the direct Boundary Element Method (BEM). In this investigation, however, the Analog Equation Method (AEM) solution is applied and the results are cross checked using the Method of Fundamental Solutions (MFS). Several micro-bars of various cross-sections are analyzed to illustrate the applicability of the developed model and to reveal the differences between the current model and an existing one which, however, contains two additional constants related to the microstructure. Moreover, useful conclusions are drawn from the micron-scale torsional response of micro-bars, giving thus a better insight in the gradient elasticity approach of the deformable bodies.

1. Introduction

In recent years a need has been raised in engineering practice to predict accurately the response of micron-scale structures, which can be either the components of microelectromechanical systems (MEMS) or various other micro-featured materials (such as foams, human bone, etc.) which show size-dependent mechanical behavior at different length scales (see e.g. Lakes, 1983). The behavior of such structures has been proven experimentally to be size dependent in metals (see e.g. Fleck et al., 1994; Poole et al., 1996) and in polymers (Lam and Chong, 1999; Chong and Lam, 1999). Thus, the utilization of strain gradient (higher order) theories containing internal material length scale parameters is inevitable. The couple stress theory is a special case of these higher-order theories in which the effects of the dilatation gradient and the deviatoric stretch gradient are assumed to be negligible. An analytic presentation of the aforementioned theories can be found in (Vardoulakis and Sulem, 1995; Exadaktylos and Vardoulakis, 2001; Tsepoura et al., 2002; Lubarda, 2003). Although, the strain gradient theories encounter the physical problem in its generality, they contain additional constants – besides the Lamé constants – which must be determined through meticulous experiments at small length scales (see, e.g. Lakes, 1995).

The work that has been done on the solution of the Saint–Venant torsion problem of elastic micro-bars - employing couple stress theories - is limited only to the work of Tong et al. (2004). In their work the simplified couple stress model of Lam et al. (2003) with three additional material length scale parameters is applied to the torsion problem. Since the dilatational strain gradients vanish identically, the particular model leads to the formulation of the torsional equation in terms of the warping function which contains only two material length scale parameters. Two formulations in terms of pseudo warping function and stress function are presented. However, the employed analytical solutions are restricted only to simple geometric shapes. That is, closed-form solutions for circular and thin-walled cross-section are presented while a series solution for rectangular micro-bars is also introduced. Moreover, the two additional constants, in this simplified couple stress model, are difficult to determine (Lam et al., 2003). Therefore, gradient elastic models of only one additional material constant are desirable.

Similar problems have been addressed for micropolar elastic cylinders in the published book by Iesan (2008) and in the

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exhaustive literature cited therein. In particular, Iesan (1982, 1986, 2007) formulated a method for the solution of Saint–Venant problems in micropolar beam with arbitrary cross-section. Detailed solution of the torsion problem for an isotropic micropolar beam with circular cross-section can also be found in the papers of Reddy and Venkatasubramanian (1976) and Gauthier and Jahsman (1975).

In this work the simplified couple stress theory of Yang et al. (2002) is developed for the solution of the Saint–Venant torsion problem of micro-bars with arbitrary shape. Yang et al. modifying the classical couple stress theory (e.g. Mindlin, 1964; Koiter, 1964) proposed a modified couple stress model in which only one material length parameter is needed to capture the size effect. This simplified couple stress theory is based on an additional equilibrium relation which forces the couple stress tensor to be symmetric. So far it has been developed for the static bending (Park and Gao, 2006) and free vibration (Kong et al., 2008) problems of a Bernoulli–Euler beam, for the static bending and free vibration problems of a Timoshenko beam (Ma et al., 2008) and for the solution of a simple shear problem (Park and Gao, 2008) after the derivation of the boundary conditions and the governing differential equation of the theory in terms of the displacement. Moreover, the static bending problem of Kirchhoff isotropic plates was studied by Tsiatas (2009) and of orthotropic plates by Tsiatas and Yiotis (2010).

The governing equilibrium equation and the pertinent boundary conditions in terms of the warping function are derived using the minimum potential energy principle. The resulting boundary value problem of the micro-bar is described by a fourth order partial differential equation, which represents the analog of a Kirchhoff plate under uniform tensile membrane force with mixed Neumann type boundary conditions. Since the fundamental solution of the equation is known, the problem could be solved using the direct BEM for plates by establishing the integral representation via the Betti's reciprocal theorem. Nevertheless, the problem is solved more efficiently using the AEM with the simple fundamental solution of the biharmonic operator and the results are cross checked using the MFS. The employed numerical method is capable to handle micro-bars with complex geometries. Numerical results are obtained and useful conclusions are drawn regarding the use of either couple stress model as well as the size effect on the torsional response of micro-bars, giving thus a better insight in the gradient elasticity approach of the deformable bodies.

2. Problem formulation

2.1. Derivation of the governing equations

In the modified couple stress theory presented by Yang et al. (2002), the strain energy density is a function of both strain tensor and the symmetric part of the curvature tensor which are conjugated with the stress tensor and the deviatoric part of the couple stress tensor. Thus, for a deformable body the strain energy density is given as

$$W = \frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \mathbf{m} : \boldsymbol{\chi}) \tag{1}$$

where the strain tensor $\boldsymbol{\varepsilon}$, the symmetric part of the curvature tensor $\boldsymbol{\chi}$, the stress tensor $\boldsymbol{\sigma}$ and the deviatoric part of the couple stress tensor $\mathbf{m}$ are defined as

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla) \tag{2a}$$

$$\boldsymbol{\chi} = \frac{1}{2}(\nabla\boldsymbol{\theta} + \boldsymbol{\theta}\nabla) \tag{2b}$$

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon} \tag{2c}$$

$$m = 2\mu l^2\boldsymbol{\chi} \tag{2d}$$

with $\mathbf{u}$ being the displacement vector, $\boldsymbol{\theta}$ is the rotation vector defined as (Yang et al., 2002)

$$\boldsymbol{\theta} = \frac{1}{2}\mathrm{curl}\,\mathbf{u} \tag{3}$$

λ, μ are the Lamé constants and l is a material length scale parameter. Note that the deviatoric part of the couple stress tensor $\mathbf{m}$ defined in Eq. (2d) is symmetric due to the symmetry of $\boldsymbol{\chi}$ given in Eq. (2b).

Thus, using Eq. (2) the strain energy density (1) takes the form

$$W = \frac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu\left(\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + l^2\boldsymbol{\chi} : \boldsymbol{\chi}\right) \tag{4}$$

From the above relation it is readily proven that not only the strain energy density is positive definite but also is a quadratic function of both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ (Grentzelou and Georgiadis, 2005).

Consider now an elastic bar of length L with arbitrary cross-section occupying the two-dimensional domain Ω of arbitrary shape in the x, y plane bounded by the curve Γ which may be piecewise smooth, i.e. it may have a finite number of corners. The cross-section is constant along the length of the bar and is twisted by moments M_t applied at its ends. According to Saint–Venant's torsion theory (e.g. Wagner and Gruttmann, 2001; Katsikadelis, 2002), the deformation of the bar consists of (a) rotations of the cross-sections about an axis passing through the *twist center* of the bar and (b) warping of the cross-sections, which is the same for all sections. Choosing the origin of the coordinate system at the twist center of an end section, the rotation at a distance z is ϑz, where ϑ is a constant expressing the rotation of a cross-section per unit length. Assuming that this rotation is small, the displacement components of an arbitrary point are (e.g. Wagner and Gruttmann, 2001; Katsikadelis, 2002)

$$u = -\vartheta zy \tag{5a}$$

$$v = \vartheta zx \tag{5b}$$

$$w = \vartheta\phi(x,y) \tag{5c}$$

where $\phi(x,y)$ is the *warping function*. Taking into account Eqs. (5) and (3) the displacement and rotation vectors of the micro-bar become, respectively,

$$\mathbf{u} = -\vartheta zy\mathbf{e}_1 + \vartheta zx\mathbf{e}_2 + \vartheta\phi(x,y)\mathbf{e}_3 \tag{6a}$$

$$\boldsymbol{\theta} = \frac{1}{2}\vartheta\left(\phi_{,y} - x\right)\mathbf{e}_1 - \frac{1}{2}\vartheta(\phi_{,x} + y)\mathbf{e}_2 + \vartheta z\mathbf{e}_3 \tag{6b}$$

Substituting Eq. (5) into Eqs. (2a) and (2b) the nonzero components of the strain and curvature tensor are written as

$$\gamma_{xz} = \vartheta(\phi_{,x} - y) \tag{7a}$$

$$\gamma_{xz} = \vartheta\left(\phi_{,y} + x\right) \tag{7b}$$

$$\chi_x = \frac{1}{2}\vartheta\left(\phi_{,xy} - 1\right) \tag{8a}$$

$$\chi_y = -\frac{1}{2}\vartheta\left(\phi_{,xy} + 1\right) \tag{8b}$$

$$\chi_z = \vartheta \tag{8c}$$

$$\chi_{xy} = \frac{1}{4}\vartheta\left(\phi_{,yy} - \phi_{,xx}\right) \tag{8d}$$

respectively. Moreover, the nonzero components of the stress (2c) and couple stress (2d) tensors, after the appropriate replacement of the Lamé constants by the modulus of elasticity E and the Poisson's ratio ν, take the following form

$$\tau_{xz} = G\vartheta(\phi_{,x} - y) \tag{9a}$$

$$\tau_{yz} = G\vartheta\left(\phi_{,y} + x\right) \tag{9b}$$

$$m_x = l^2 G\vartheta\left(\phi_{,xy} - 1\right) \tag{10a}$$

$$m_y = -l^2 G\vartheta\left(\phi_{,xy} + 1\right) \tag{10b}$$

$$m_z = 2l^2 G\vartheta \tag{10c}$$

$$m_{xy} = \frac{1}{2}l^2 G\vartheta\left(\phi_{,yy} - \phi_{,xx}\right) \tag{10d}$$

where $G = E/2(1+\nu)$ is the shear modulus.

In the absence of body force and body couple and taking into account that the cylindrical surface of the micro-bar is traction and surface couple free, the first variation of the total potential energy takes the form (Tsiatas, 2009)

$$\delta\Pi = \frac{1}{2}\int_V (\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} + \mathbf{m} : \delta\boldsymbol{\chi})\mathrm{d}V \tag{11}$$

and using Eqs. (7)–(10) yields

$$\delta\Pi = \vartheta\int_\Omega \left[\tau_{xz}\delta\phi_{,x} + \tau_{yz}\delta\phi_{,y} - \frac{1}{2}m_{xy}\delta\phi_{,xx} + \frac{1}{2}m_{xy}\delta\phi_{,yy} + \frac{1}{2}(m_x - m_y)\delta\phi_{,xy}\right]\mathrm{d}\Omega \tag{12}$$

which, after the transformation of the domain integral using twice the divergence theorem of Gauss, becomes

$$\begin{aligned}\delta\Pi = &-\vartheta\int_\Omega\left[\tau_{xz,x} + \tau_{yz,y} + \frac{1}{2}(m_{xy,xx} - m_{xy,yy} + m_{y,xy} - m_{x,xy})\right]\delta\phi\,\mathrm{d}\Omega \\ &+\vartheta\int_\Gamma\left[\tau_{xz}n_x + \tau_{yz}n_y + \frac{1}{2}(m_{xy,x}n_x - m_{xy,y}n_y + m_{y,y}n_x - m_{x,x}n_y)\right] \\ &\times\delta\phi\,\mathrm{d}s - \vartheta\frac{1}{2}\int_\Gamma m_{n\nu}\delta\phi_{,n}\,\mathrm{d}s + \vartheta\frac{1}{2}\int_\Gamma m_n\delta\phi_{,\nu}\,\mathrm{d}s\end{aligned} \tag{13}$$

where

$$m_n = m_x n_x^2 + m_y n_y^2 + 2m_{xy}n_x n_y \tag{14a}$$

$$m_{n\nu} = m_{xy}\left(n_x^2 - n_y^2\right) + (m_y - m_x)n_x n_y \tag{14b}$$

are the stress resultants; $\mathbf{n}(n_x, n_y)$ and $\mathbf{t}(-n_y, n_x)$ are the unit (outward) vector normal to the boundary and the unit tangent to the boundary, respectively, ($n_x = \cos a$, $n_y = \sin a$ with $a = \measuredangle x, \mathbf{n}$).

The first line integral in Eq. (13) represents a line force term along the boundary (the respective shearing force term in the plate bending theory e.g. Katsikadelis and Armenakas, 1989). The last integral in the same equation represents also a line force term and must be converted in order to be inserted into the first line integral. Noting that $\phi_{,t} = \phi_{,s}$ the integration by parts along the boundary Γ of the aforementioned integral gives

$$\begin{aligned}\int_\Gamma m_n\delta\phi_{,s}\,\mathrm{d}s &= \int_\Gamma (m_n\delta\phi)_s\,\mathrm{d}s - \int_\Gamma m_{n,s}\delta\phi\,\mathrm{d}s \\ &= \sum_k [m_n]_k\delta\phi - \int_\Gamma m_{n,s}\delta\phi\,\mathrm{d}s\end{aligned} \tag{15}$$

where $[m_n]_k$ is the jump of discontinuity of the twisting moment at the k-th corner. Thus, Eq. (13) becomes

$$\begin{aligned}\delta\Pi = &-\vartheta\int_\Omega\left[\tau_{xz,x} + \tau_{yz,y} + \frac{1}{2}(m_{xy,xx} - m_{xy,yy} + m_{y,xy} - m_{x,xy})\right]\delta\phi\,\mathrm{d}\Omega \\ &+\vartheta\int_\Gamma\left[\tau_{xz}n_x + \tau_{yz}n_y + \frac{1}{2}(m_{xy,x}n_x - m_{xy,y}n_y + m_{y,y}n_x - m_{x,x}n_y)\right. \\ &\left.-\frac{1}{2}m_{n,s}\right]\delta\phi\,\mathrm{d}s - \vartheta\frac{1}{2}\int_\Gamma m_{nt}\delta\phi_{,n}\,\mathrm{d}s + \vartheta\frac{1}{2}\sum_k [m_n]_k\delta\phi\end{aligned} \tag{16}$$

By applying the principle of total minimum potential energy, i.e., $\delta\Pi = 0$ for the stable equilibrium and the fundamental lemma of the calculus of variation (e.g. Reddy, 1999) the governing equilibrium differential equation of the micro-bar is obtained as

$$\tau_{xz,x} + \tau_{yz,y} + \frac{1}{2}(m_{xy,xx} - m_{xy,yy} + m_{y,xy} - m_{x,xy}) = 0 \quad \text{in } \Omega \tag{17}$$

together with the boundary conditions

$$\tau_{xz}n_x + \tau_{yz}n_y + \frac{1}{2}(m_{xy,x}n_x - m_{xy,y}n_y + m_{y,y}n_x - m_{x,x}n_y) - \frac{1}{2}m_{n,s} = 0 \tag{18a}$$

$$m_{nt} = 0 \tag{18b}$$

on Γ and

$$\sum_k [m_n]_k = 0 \tag{18c}$$

at the k-th corner.

Eqs. (17), (18a) and (18b) can be also verified by substituting Eqs. (7)–(10) into the general equilibrium equations

$$\mathrm{div}\boldsymbol{\sigma} + \frac{1}{2}\,\mathrm{curl}(\mathrm{div}\,\mathbf{m} + \mathbf{c}) + \mathbf{b} = \mathbf{0}, \quad \text{in } \Omega \tag{19}$$

produced by Park and Gao (2008), together with the boundary conditions

$$\boldsymbol{\sigma}\mathbf{n} + \frac{1}{2}\mathbf{n}\times[\mathrm{div}\,\mathbf{m} - \nabla(\mathbf{m} : \mathbf{n}\otimes\mathbf{n}) + \mathbf{c}] = \tilde{\mathbf{T}} - \frac{1}{2}\mathbf{n}\times\nabla\left(\tilde{\mathbf{S}}\cdot\mathbf{n}\right) \tag{20a}$$

$$\mathbf{mn} - (\mathbf{m} : \mathbf{n}\otimes\mathbf{n})\mathbf{n} = \tilde{\mathbf{S}} - \left(\tilde{\mathbf{S}}\cdot\mathbf{n}\right)\mathbf{n} \tag{20b}$$

on Γ, of a three-dimensional deformable body for the modified couple stress theory of Yang et al. (2002), in the absence of body force, body couple, traction and surface couple. In Eq. (20) and in whichever follows, the tilde over a symbol represents prescribed quantity.

Substituting Eqs. (9) and (10) into Eqs. (17) and (18) yields the governing equation of the micro-bar in terms of the warping function

$$\frac{l^2}{4}\nabla^4\phi - \nabla^2\phi = 0, \quad \text{in } \Omega \tag{21}$$

and the boundary conditions

$$\phi_{,n} - \frac{l^2}{4}\left[\nabla^2\phi_{,n} + 2(\phi_{,nt})_{,s}\right] = yn_x - xn_y \tag{22a}$$

$$\phi_{,tt} - \phi_{,nn} = 0 \tag{22b}$$

on Γ.

On the end cross-sections $z = 0$ and $z = L$, it is $n_x = n_y = 0$ and $n_z = 1$. Thus, the nonzero boundary conditions (20) are

$$T_x = \tau_{xz} - \frac{1}{2}(m_{xy,x} + m_{y,y}) = G\vartheta\left(\phi_{,x} - y + \frac{l^2}{4}\nabla^2\phi_{,x}\right) \tag{23a}$$

$$T_y = \tau_{yz} + \frac{1}{2}(m_{x,x} + m_{xy,y}) = G\vartheta\left(\phi_{,y} + x + \frac{l^2}{4}\nabla^2\phi_{,y}\right) \tag{23b}$$

Along the boundary Γ of the surface is also present a line force with components

$$\begin{aligned} g_x &= \frac{1}{2}n_y(m_y - m_x) + n_x m_{xy} \\ &= G\vartheta\left[-l^2\left(n_y\phi_{,xy} - n_x\phi_{,yy}\right) - \frac{l^2}{2}n_x\nabla^2\phi\right] \end{aligned} \tag{24a}$$

$$\begin{aligned} g_y &= \frac{1}{2}n_x(m_y - m_x) - n_y m_{xy} \\ &= G\vartheta\left[-l^2\left(n_x\phi_{,xy} - n_y\phi_{,xx}\right) - \frac{l^2}{2}n_y\nabla^2\phi\right] \end{aligned} \tag{24b}$$

in the x and y direction, respectively.

We can readily prove that the stress resultants of the tractions (23) and line forces (24) vanish. Namely,

$$\int_\Omega T_x \, \mathrm{d}\Omega + \int_\Gamma g_x \, \mathrm{d}s = 0 \tag{25a}$$

$$\int_\Omega T_y \, \mathrm{d}\Omega + \int_\Gamma g_y \, \mathrm{d}s = 0 \tag{25b}$$

The moment resultant on the cross-section is going to be

$$M_t = \int_\Omega (xT_y - yT_x)\mathrm{d}\Omega + \int_\Gamma \left(xg_y - yg_x\right)\mathrm{d}s \tag{26}$$

which, after the substitution of Eqs. (23) and (24) takes the form

$$M_t = G\vartheta\int_\Omega \left(x^2 + y^2 + x\phi_{,y} - y\phi_{,x}\right)\mathrm{d}\Omega + 3l^2 G\vartheta \tag{27}$$

Setting

$$I_t = \int_\Omega \left(x^2 + y^2 + x\phi_{,y} - y\phi_{,x}\right)\mathrm{d}\Omega + 3l^2 \tag{28}$$

we arrive at

$$M_t = G\vartheta I_t \tag{29}$$

The torsional constant I_t *does not* depend only on the shape of the cross-section, as it happens in the classical Saint-Venant theory, but it depends also on the microstructure of the micro-bar.

The domain integral in Eq. (28) can be converted into a boundary line integral (Katsikadelis, 2002). Thus, Eq. (28) finally becomes

$$I_t = \int_\Gamma \left[\left(xy^2 - y\phi\right)n_x + \left(yx^2 + x\phi\right)n_y\right]\mathrm{d}s + 3l^2 \tag{30}$$

2.2. The plate analog

The equation of a plate with bending stiffness D subjected to a uniform tensile membrane force N in absence of external load, is written as

$$D\nabla^4 w - N\nabla^2 w = 0 \quad \text{in } \Omega \tag{31}$$

Further, we consider the natural boundary conditions

$$Nw_{,n} + V_n(w) = yn_x - xn_y \tag{32a}$$

$$M_n(w) = 0 \tag{32b}$$

on Γ, where V_n and M_n are differential operators defined as

$$V_n = -D\left[\frac{\partial}{\partial n}\nabla^2 - (\nu - 1)\frac{\partial}{\partial s}\left(\frac{\partial^2}{\partial n \partial t}\right)\right] \tag{33a}$$

$$M_n = -D\left[\nabla^2 + (\nu - 1)\frac{\partial^2}{\partial t^2}\right] \tag{33b}$$

which represent the effective shear force and bending moment, respectively, on the boundary.

It is apparent that Eqs. (21) and (22) can be obtained from Eqs. (31) and (32) for $w = \phi, N = 1, D = l^2/4$ and $\nu = -1$. Thus, in this case the warping function represents the deflection of a plate subjected to a uniform tensile membrane force $N = 1$ with bending stiffness $D = l^2/4$ and Poisson's ratio $\nu = -1$ in the absence of external load. Note that, for $D = 0$ it is $V_n = M_n = 0$ and Eqs. (31) and (32) give the membrane analog for the classical Saint–Venant problem introduced by Prandtl and others. It should be mentioned that the deflection surface is not uniquely determined, since the boundary conditions permit a rigid body motion. This, however, does not influence the deformation of the cross-section (Katsikadelis, 2002).

2.3. The modified couple stress torsion model of Tong et al.

In the work of Tong et al. (2004), the couple stress model of Lam et al. (2003) - with three additional material length scale parameters - is applied to the Saint–Venant torsion problem. Since the dilatational strain gradients vanish identically, the torsion model contains only two material length scale parameters, namely l_1 and l_2. The governing equation of the micro-bar in terms of the warping function is

$$\left(\frac{8l_1^2}{15} + \frac{l_2^2}{4}\right)\nabla^4\phi - \nabla^2\phi = 0 \quad \text{in } \Omega \tag{34}$$

and the boundary conditions are

$$\phi_{,n} - \left(\frac{8l_1^2}{15} + \frac{l_2^2}{4}\right)\nabla^2\phi_{,n} - \left(\frac{2l_1^2}{3} + \frac{l_2^2}{2}\right)(\phi_{,nt})_s = yn_x - xn_y \tag{35a}$$

$$\left(\frac{8l_1^2}{15} + \frac{l_2^2}{4}\right)\nabla^2\phi - \left(\frac{2l_1^2}{3} + \frac{l_2^2}{2}\right)\phi_{,tt} = 0 \tag{35b}$$

on Γ. The boundary tractions on the end cross-sections are

$$T_x = \tau_{xz} - \frac{1}{2}(m_{xy,x} + m_{y,y}) = G\vartheta\left[\phi_{,x} - y - \left(\frac{16l_1^2}{15} - \frac{l_2^2}{4}\right)\nabla^2\phi_{,x}\right] \tag{36a}$$

$$T_y = \tau_{yz} + \frac{1}{2}(m_{x,x} + m_{xy,y}) = G\vartheta\left[\phi_{,y} + x - \left(\frac{16l_1^2}{15} - \frac{l_2^2}{4}\right)\nabla^2\phi_{,y}\right] \tag{36b}$$

while the moment resultant on the cross-section takes the form

$$M_t = G\vartheta\int_{\Omega}\left(x^2 + y^2 + x\phi_{,y} - y\phi_{,x}\right)d\Omega + 3l_2^2 G\vartheta \tag{37}$$

Note that setting $l_1 \to 0$ and $l_2 = l$ in the above equations yield Eqs. (21)–(23) and (27) of the proposed model.

Eqs. (34) and (35) can be also obtained from Eqs. (31) and (32) for $w = \phi$, $N = 1$ and

$$D = \frac{8l_1^2}{15} + \frac{l_2^2}{4} \tag{38a}$$

$$\nu = 1 - \left(\frac{2l_1^2}{3} + \frac{l_2^2}{2}\right) \Big/ \left(\frac{8l_1^2}{15} + \frac{l_2^2}{4}\right) \tag{38b}$$

3. The numerical solution

Using the Betti's reciprocal theorem for the plate equation, in the absence of external load, we obtain the integral representation of the solution as (Katsikadelis and Babouskos, 2009)

$$\begin{aligned} w(P) = & \int_{\Gamma}\left[vV_n(w) - wV_n(v) - \frac{\partial v}{\partial n}M_n(w) + \frac{\partial w}{\partial n}M_n(v)\right]ds \\ & - \sum_k \left(v\|T(w)\|_k - w\|T(v)\|_k\right) \end{aligned} \tag{39}$$

where $P : \{x, y\} \in \Omega$ and v is the fundamental solution of Eq. (31), i.e. a singular particular solution of the following equation

$$\nabla^4 v - \mu^2\nabla^2 v = \delta(P - Q) \tag{40}$$

given as

$$v = \frac{1}{2\pi\mu^2}[K_0(\mu r) - \ln r] \tag{41}$$

with K_0 being the zero-order modified Bessel function of the second kind and $\mu^2 = N/D$. T is a differential operator defined as

$$T = D(1 - \nu)\frac{\partial^2}{\partial n \partial t} \tag{42}$$

which represents the twisting moment M_{nt} along the boundary and $\|T(w)\|_k$ its jump of discontinuity at the k-th corner.

Obviously, the boundary integral equations will result for $P \to p \in \Gamma$. Thus, the warping function can be established by developing the direct BEM.

However, in order to avoid rather complicated computations of singular integrals, the problem is solved using the AEM (Tsiatas and Yiotis, 2010) discretization which employs the simple fundamental solution

$$v = \frac{1}{8\pi D}r^2 \ln r \tag{43}$$

of the biharmonic equation. The results are cross checked using the MFS as it was applied for plates (Tsiatas, 2009).

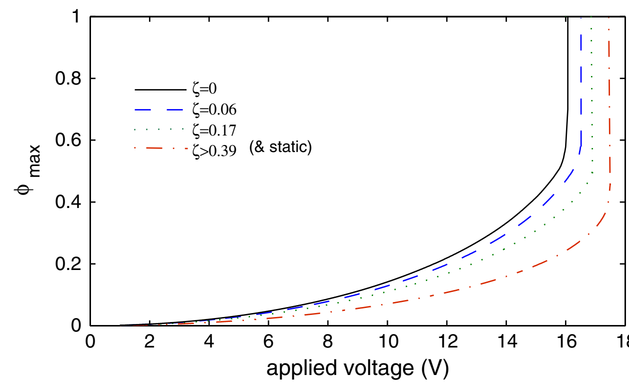

Fig. 1. Normalized torsional constant versus the material length scale parameter of the square micro-bar. Tong et al. model: $l_1 = l_2$; proposed model: $l_1 = 0$, $l_2 = l$.

4. Numerical examples

On the base of the procedure described in previous section a FORTRAN program has been written for establishing the torsional response of the micro-bars. In the MFS the source points are placed equally on a virtual boundary – outside the domain – at a distance 20% greater than that of the actual one.

4.1. Square micro-bar

For reasons of comparisons a square micro-bar ($a/b = 1$) was first investigated employing both couple stress models. In Fig. 1 is depicted the normalized torsional constant I_t/I_t^c (I_t^c is the torsional constant of the classical Saint–Venant theory) versus the material length scale parameter l_2. The results from the AEM and MFS solution employing the Tong et al. model ($l_1 = l_2$) are found to be in excellent agreement with that obtained from their analytical solution. We can also observe that the torsional constant estimated by the proposed one-parameter model ($l_1 \to 0$, $l_2 = l$) is smaller as the material length scale parameter increases to the value of $l_2 = 0.3$, while, for greater values the difference between the two models becomes negligible. The presented results indicate that the torsional constant of the bar increases nonlinearly with the increase of l_2 in both models. Moreover, in Fig. 2 is depicted the warping surface for the case $l_1 = l_2 = 0.3$.

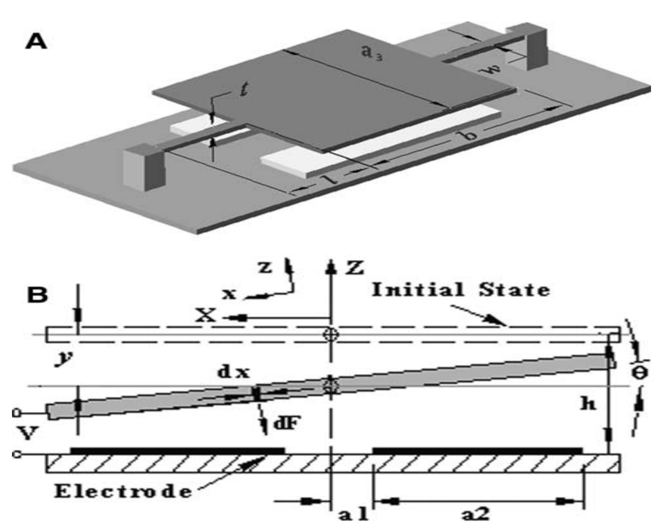

Fig. 2. Warping surface of the square micro-bar.

Table 1
Torsional constant of the rectangular micro-bar.

l_2	$a/b = 0.8$		$a/b = 1.2$	
	Tong et al. model	Proposed model	Tong et al. model	Proposed model
0.0	0.13724	0.13724	0.13838	0.13838
0.1	0.16153	0.15017	0.20210	0.16420
0.2	0.27064	0.24767	0.27822	0.25199
0.3	0.42939	0.40577	0.43386	0.40986
0.4	0.64353	0.62311	0.64604	0.62834

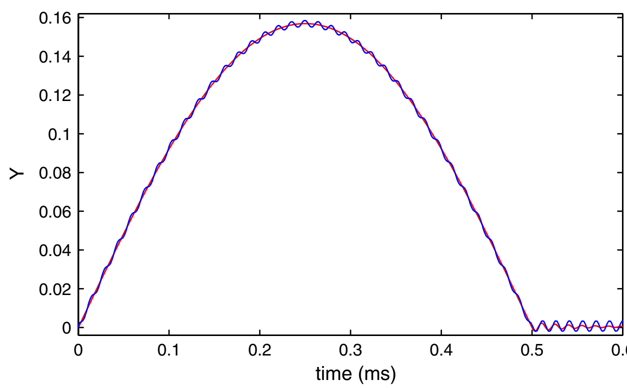

Fig. 3. Normalized torsional constant versus the material length scale parameter of the rectangular micro-bars. Tong et al. model: $l_1 = l_2$; proposed model: $l_1 = 0$, $l_2 = l$.

4.2. Rectangular micro-bars

Afterwards, two micro-bars with rectangular cross-sections of width a and height b have been analyzed ($N = 100$) in order to examine the influence of the micro-bar shape on the torsional constant. The rectangular dimensions ($a/b = 0.894/1.118 = 0.8$ and $a/b = 1.095/0.913 = 1.2$) were chosen such as the area of the cross-section was kept fixed $A = ab = 1$. In Table 1 results for the torsional constant are presented for both models and aspect ratios. In Fig. 3 is also depicted the normalized torsional constant I_t/I_t^c versus the material length scale parameter l_2. From this figure we

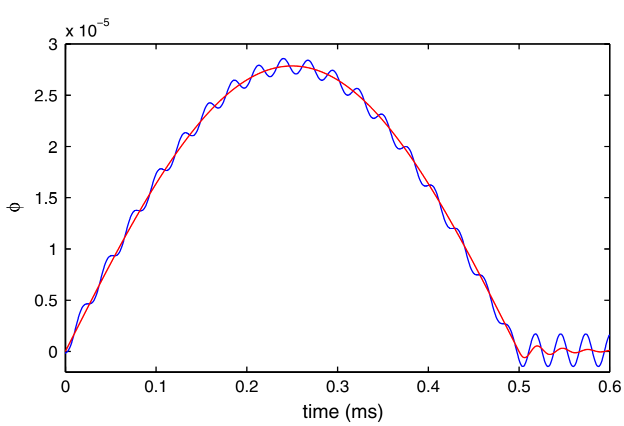

Fig. 4. Normalized torsional constant versus the material length scale parameter of the circular and elliptical micro-bars. Tong et al. model: $l_1 = l_2$; proposed model: $l_1 = 0$, $l_2 = l$.

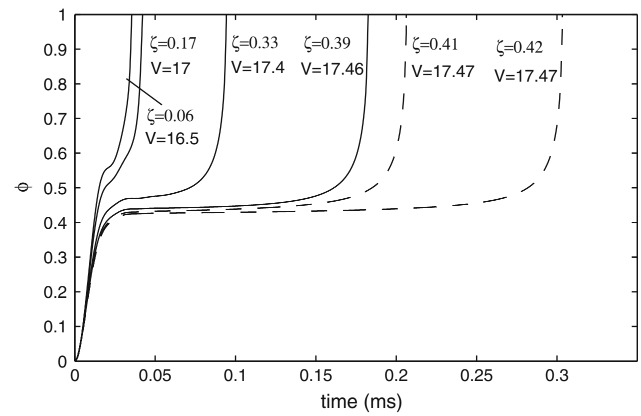

Fig. 5. Contours of the warping surface of the elliptical micro-bar.

can observe that, unlike the case of the square micro-bar, the torsional constant estimated by the proposed one-parameter model is always smaller as the material length scale parameter increases. Moreover, from the same figure can be pointed out that, beyond the value of $l_2 = 0.3$ the normalized torsional constant do not depend on the dimensional aspect ratio in both models.

4.3. Circular and elliptical micro-bars

In order to investigate the micron-scale torsional response on curved cross-sections a circular and an elliptical micro-bar have been analyzed ($N = 100$). In Fig. 4 the normalized torsional constant I_t/I_t^c versus the material length scale parameter l_2 is shown for an elliptical cross-section with semi axes $a = 1, b = 1.2$ and a circular cross-section of radius $r = a = b = 1$. For the circular cross-section the results from both models are identically the same while for the elliptical one the difference between the two models is very small. Moreover, the contours of the warping surface for the proposed model ($l_2 = 0.4$) are depicted in Fig. 5.

5. Conclusions

In this paper the Saint–Venant torsion problem of micro-bars of arbitrary cross-section was solved. The proposed model is derived from the modified couple stress theory of Yang et al. (2002) and has only one material length scale parameter. The governing equilibrium equation and the associated boundary conditions of the micro-bar are derived in terms of the warping function using the principle of minimum potential energy. The resulting boundary value problem is of the fourth order and it is solved using the Analog Equation Method (AEM), while the results are cross checked using the Method of Fundamental Solutions (MFS). The main conclusions that can be drawn from this investigation are summarized as:

- Both Saint–Venant torsion models are described by a fourth order partial differential equation representing the analog of a Kirchhoff plate having the shape of the cross-section and subjected to a uniform tensile membrane force with mixed Neumann boundary conditions.

- The present model is derived from the modified couple stress theory of Yang et al. and has only one material length scale parameter, which, indeed, is easier to determine as compared to Tong et al. model which contains two additional constants related to the microstructure of the material.
- The obtained results from the AEM and MFS solution are found to be in excellent agreement as compared with that obtained from analytical solution.
- In all examples the torsional constant of the micro-bar increases nonlinearly with the increase of the material length scale l_2 in both models. As well as, the torsional constant estimated by the proposed one-parameter model is always smaller compared to the Yang et al. model.
- For the examined micro-bar with square cross-section, the torsional constant estimated by the proposed one-parameter model is smaller as the material length scale parameter increases to the value of $l_2 = 0.3$, while, for greater values the difference between the two models becomes negligible.
- For the examined micro-bars with rectangular cross-section, the normalized torsional constant beyond the value of $l_2 = 0.3$ do not depend on the dimensional aspect ratio in both models.
- The normalized torsional constant from both models is identically the same for micro-bars with circular cross-section while for those with elliptical one the difference between the two models is very small.

References

Chong, A.C.M., Lam, D.C.C., 1999. Strain gradient plasticity effect in indentation hardness of polymers. J. Mater. Res. 14, 4103–4110.

Exadaktylos, G.E., Vardoulakis, I., 2001. Microstructure in linear elasticity and scale effects: a reconsideration of basic rock mechanics and rock fracture mechanics. Tectonophysics 335, 81–109.

Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W., 1994. Strain gradient plasticity: theory and experiments. Acta Metall. Mater. 42, 475–487.

Gauthier, R.D., Jahsman, W.E., 1975. A quest for micropolar elastic constants. J. Appl. Mech. 42, 369–374.

Grentzelou, C.G., Georgiadis, H.G., 2005. Uniqueness for plane crack problems in dipolar gradient elasticity and in couple-stress elasticity. Int. J. Solids Struct. 42, 6226–6244.

Iesan, D., 1982. Saint–Venant's problem in micropolar elasticity. In: Brulin, O., Hsieh, R.K.T. (Eds.), Mechanics of Micropolar Media. World Scientific, Singapore, pp. 281–390.

Iesan, D., 1986. Generalized twist for the torsion of micropolar cylinders. Meccanica 21, 94–96.

Iesan, D., 2007. Generalized torsion of elastic cylinders with microstructure. Note di Matematica 27 (2), 121–130.

Iesan, D., 2008. Classical and generalized models of elastic rods. In: Gao, D., Ogden, R.W. (Eds.), CRC Series: Modern Mechanics and Mathematics. CRC Press.

Katsikadelis, J.T., 2002. Boundary Elements: Theory and Applications. Elsevier, Amsterdam-London.

Katsikadelis, J.T., Armenakas, A.E., 1989. New boundary equation solution to the plate problem. J. Appl. Mech.-T. ASME 56, 364–374.

Katsikadelis, J.T., Babouskos, N.G., 2009. Nonlinear flutter instability of thin damped plates. A solution by the AEM. J. Mech. Mater. Struct. 4, 1395–1414.

Koiter, W.T., 1964. Couple stresses in the theory of elasticity, I and II. Proc. K. Ned. Akad. Wet. (B) 67, 17–44.

Kong, S., Zhou, S., Nie, Z., Wang, K., 2008. The size-dependent natural frequency of Bernoulli–Euler micro-beams. Int. J. Eng. Sci. 46, 427–437.

Lakes, R.S., 1983. Size effects and micromechanics of a porous solid. J. Mater. Sci. 18, 2572–2580.

Lakes, R.S., 1995. Experimental methods for study of cosserat elastic solids and other generalized elastic continua. In: Mühlhaus, H. (Ed.), Continuum Models for Materials with Microstructure. Wiley, pp. 1–25.

Lam, D.C.C., Chong, A.C.M., 1999. Indentation model and strain gradient plasticity law for glassy polymers. J. Mater. Res. 14, 3784–3788.

Lam, D.C.C., Yang, F., Chong, A.C.M., Wang, J., Tong, P., 2003. Experiments and theory in strain gradient elasticity. J. Mech. Phys. Solids 51, 1477–1508.

Lubarda, V.A., 2003. The effects of couple stresses on dislocation strain energy. Int. J. Solids Struct. 40, 3807–3826.

Ma, H.M., Gao, X.-L., Reddy, J.N., 2008. A microstructure-dependent Timoshenko beam model based on a modified couple stress theory. J. Mech. Phys. Solids 56, 3379–3391.

Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78.

Park, S.K., Gao, X.-L., 2006. Bernoulli–Euler beam model based on a modified couple stress theory. J. Micromech. Microeng. 16, 2355–2359.

Park, S.K., Gao, X.-L., 2008. Variational formulation of a modified couple stress theory and its application to a simple shear problem. Z. Angew. Math. Phys. 59, 904–917.

Poole, W.J., Ashby, M.F., Fleck, N.A., 1996. Micro-hardness of annealed and work-hardened copper polycrystals. Scripta Metall. Mater. 34 (4), 559–564.

Reddy, G.V.K., Venkatasubramanian, N.K., 1976. Saint–Venant's problem for a micropolar elastic circular cylinder. Int. J. Eng. Sci. 14, 1047–1057.

Reddy, J.N., 1999. Theory and Analysis of Elastic Plates. Taylor and Francis, Philadelphia.

Tong, P., Yang, F., Lam, D.C.C., Wang, J., 2004. Size effects of hair-sized structures - torsion. Key Eng. Mater. 261–263, 11–22.

Tsepoura, K., Papargyri-Beskou, S., Polyzos, D., Beskos, D.E., 2002. Static and dynamic analysis of gradient elastic bars in tension. Arch. Appl. Mech. 72, 483–497.

Tsiatas, G.C., 2009. A new Kirchhoff plate model based on a modified couple stress theory. Int. J. Solids Struct. 46, 2757–2764.

Tsiatas, G.C., Yiotis, A.J., 2010. A microstructure-dependent orthotropic plate model based on a modified couple stress theory. In: Sapountzakis, E. (Ed.), Recent Developments in Boundary Element Methods. A Volume to Honour Professor John T. Katsikadelis. WIT Press, Southampton, pp. 295–308.

Vardoulakis, I., Sulem, J., 1995. Bifurcation Analysis in Geomechanics. Blackie/Chapman and Hall, London.

Wagner, W., Gruttmann, F., 2001. Finite element analysis of Saint–Venant torsion problem with exact integration of the elastic-plastic constitutive equations. Comput. Methods Appl. Mech. Eng. 190, 3831–3848.

Yang, F., Chong, A.C.M., Lam, D.C.C., Tong, P., 2002. Couple stress based strain gradient theory for elasticity. Int. J. Solids Struct. 39, 2731–2743.