Pair copula constructions to determine the dependence structure of Treasury bond yields

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Keywords
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Abstract
We estimated the dependence structure of US Treasury bonds through a pair copula construction. As a result, we verified that the variability of the yields decreases with a longer time of maturity of the bond. The yields presented strong dependence with past values, strongly positive bivariate associations between the daily variations, and prevalence of the Student’s t copula in the relationships between the bonds. Furthermore, in tail associations, we identified relevant values in most of the relationships, which highlights the importance of risk management in the context of bonds diversification.

Introduction
Since the introduction of the mathematical theory of portfolio selection and of the capital asset pricing model (CAPM), the issue of dependence has always been of fundamental importance to financial economics. In the context of international diversification, there is a need to minimise the risk of specific assets (such as stocks and Treasury bonds) through optimal allocation of resources. Many studies have used a statistical model which is able to measure the temporal dependence between stocks and Treasury bonds: Campbell and Ammer (1993) apply a vector autoregressive (VAR) system in AMEX and NYSE stocks and US Treasury bonds, but they do not analyse the effect of the volatility of the relationship. Li (2002) and Kim, Moshirian, and Wu (2006) estimate a bivariate generalised autoregressive conditional heteroscedasticity (GARCH) model and bivariate exponential GARCH with t-distribution and verify important implications in stock-bonds correlation. However, Cappiello, Engle, and Sheppard (2006), and Li and Zou (2008) extend the asymmetric and multivariate approach with dynamic conditional correlation (DCC) GARCH.

Traditionally, correlation is used to describe the dependence between random variables, but recent studies, such as that conducted by Embrechts, Lindskog, and McNeil (2003), have ascertained the superiority of copulas to model dependence. Copulas offer much more flexibility than the correlation...
approach because a copula function can deal with non-linearity, asymmetry, serial dependence and also the well-known heavy-tails of financial assets’ marginal and joint probability distribution. In studies of Treasury bonds, Junker, Szimayer, and Wagner (2006) apply the normal copula model in US Treasury monthly bonds, confirming the importance of this approach in considering tail dependence and symmetry. Lee, Kim, and Kim (2011) apply Archimedean copulas in testing the forecast accuracy of bonds, where the Student’s t and Clayton mixture copula outperforms the other copulas considered.

A copula is a function that links univariate marginals to their multivariate distribution. Since it is always possible to map any vector of random variables into a vector with uniform margins, we are able to split the margins of that vector, which is the copula. Although the literature on copulas is consistent, the great part of the research is still limited to the bivariate case. Thus, constructing higher dimensional copulas is the natural next step, but this is not an easy task. Apart from the multivariate Gaussian and Student (see work in stock-bonds structure dependence of government bonds, where the Student’s t and Clayton mixture copula outperforms the other copulas considered.

The developments in this area tend to hierarchical, copula-based structures. It is possible that the most promising of these is the pair copula construction (PCC). Originally proposed by Joe (1996), it has been further discussed and explored in the literature for questions of inference and simulation. The PCC is based on a decomposition of a multivariate density into bivariate copula densities, of which some are dependency structures of unconditioned bivariate distributions, and the rest are dependency structures of conditional bivariate distributions. Applications to financial data have shown that these vine-PCC models outperform other multivariate copula models in predicting log-returns of equity portfolios. Min and Czado (2010) present a PCC copula model in daily returns from January 1, 1999 to July 8, 2003 of the Norwegian stock index, the MSCI world stock index, the Norwegian bond index and the SSBWG hedged bond index, and they verify a stronger dependence between international bonds and stocks, international and Norwegian stocks, and Norwegian stocks and bonds, but they observe that the Norwegian bond index does not depend on the MSCI world stock index if the Norwegian stock index is given. In this context, this paper posed the question: What would the dependence structure of Treasury bonds be in relation to their maturity?

To answer this question, this paper aims to estimate the dependence structure between Treasury bonds through a PCC. To that effect, we collected daily data from Treasury bonds of the US government for 1-, 2-, 3-, 5-, 7- and 10-years of maturity, which were the most sought after by investors in order to obtain truly risk free assets. The estimated structure allows the calculation of the non-linear absolute and tail dependencies of each bivariate relationship between the bonds, isolating the effect of the other. It is also possible to verify which bond has more dependence with all the others, and to identify the “leading” Treasury bond.

The paper is structured as follows: The second section briefly presents the background of copulas and PCC; the third section presents the material and methods of the study, describing the data and the procedures used to achieve the objective of the paper; the fourth section presents the results obtained and the discussion; and the fifth section contains the conclusions of the paper; the appendix introduces the copula families utilised in this study.

Background

This section is subdivided into: i) Copula methods, which briefly defines this class of functions and describes its properties; this sub section also contains a literature review; ii) Pair copula construction, which succinctly describes the concepts of this structure.

Copula methods

Dependence between random variables can be modelled by copulas. A copula returns the joint probability of events as a function of the marginal probabilities of each event. This makes copulas attractive, as the univariate marginal behaviour of random variables can be modelled separately from their dependence (Kojadinovic & Yan, 2010).

The concept of copula was introduced by Sklar (1959). However, it was only recently that its applications became clear. A detailed treatment of copulas as well as of their relationship to concepts of dependence is given by Joe (1997) and Nelsen (2006). A review of the applications of copulas to finance can be found in Embrechts et al. (2003) and in Cherubini, Luciano, and Vecchiato (2004).

To facilitate our understanding of the concept we restrict our attention to the bivariate case. The extensions to the n-dimensional case are straightforward. A function $C: [0,1]^2 \rightarrow [0,1]$ is a copula if, for $0 \leq x \leq 1$ and $x_1 \leq x_2$, $y_1 \leq y_2$, $(x_1, y_1), (x_2, y_2) \in [0,1]^2$, it fulfills the following properties:

$$C(x, 1) = C(1, x) = x, \quad C(x, 0) = C(0, x) = 0.$$  \hfill (1)

$$C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0.$$  \hfill (2)

Property (1) means uniformity of the margins, while (2), the n-increasing property means that $P(x_1 \leq x \leq x_2, y_1 \leq Y \leq y_2) \geq 0$ for $(X, Y)$ with distribution function $C$.

In Sklar’s seminal paper (1959), it was demonstrated that a copula is linked with a distribution function and its marginal distributions. This important theorem states:

(i) Let $C$ be a copula and $F_1$ and $F_2$ univariate distribution functions. Then (3) defines a distribution function $F$ with marginals $F_1$ and $F_2$.

$$F(x, y) = C(F_1(x), F_2(y)), \quad (x, y) \in \mathbb{R}^2.$$  \hfill (3)

(ii) For a two-dimensional distribution function $F$ with marginal $F_1$ and $F_2$, there is a copula $C$ satisfying (3). This is unique if $F_1$ and $F_2$ are continuous and then, for every
\[ (u, v) \in [0, 1]^2: \]
\[ C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)). \]  

(4)

In (4), \( F_1^{-1} \) and \( F_2^{-1} \) denote the generalised left continuous inverses of \( F_1 \) and \( F_2 \). Regarding the estimation, the dominant methods are the traditional maximum likelihood (ML), the pseudo-maximum likelihood (PML), proposed by Genest, Ghoudi, and Rivest (1995), and the inversion of dependence measures such as Spearman’s rho and Kendall’s tau. Chen and Fan (2006b) developed an extension of the pseudo-maximum likelihood of Markovian time series.

However, as Frees and Valdez (1998) note, it is not always possible to identify the copula. According to Berrada, Dupuis, Jacquier, Papageorgiou, and Rémillard (2006), for many financial applications, the problem is not in using a given multivariate distribution but in finding a convenient distribution to describe some stylised facts, for example the relationships between different asset returns. Genest et al. (2009) present an overview of the goodness of fit and selection issues of copula families.

Since copulas are linked to the dependence structure, they must be related to dependence measures. We present here the calculation procedures, adapted from Cherubini, Gobbi, Mulinacci, and Romagnoli (2012), of the most representative dependence measures for financial purposes. Given the estimated bivariate copula \( C \), the lower and upper tail dependence are represented by formulations (5) and (6), respectively. The absolute dependence calculated with Kendall’s tau through the conversion of the bivariate copula is exposed in formulation (7).

\[ \lambda_L = \lim_{u \to 0} \frac{C(u, u)}{u} \]  

(5)

\[ \lambda_U = \lim_{u \to 1} 1 - 2u + \frac{C(u, u)}{1-u} \]  

(6)

\[ \tau(x, y) = 4 \int_0^1 \int_0^1 (C(u, v)) \, dC(u, v) - 1. \]  

(7)

Regarding the literature on copula methods, it is noteworthy that there was a significant growth in the number of applications of this technique in the last few years. With reference to time series, one of the most appealing approaches is the time-varying copulas, which consist of the change of the shape and parameters of the estimated copula families along time. Some of the most structured proposals on the topic are the works of Chen and Fan (2006a, 2006b) and Patton (2006, 2011). As a financial application of dynamic copulas, we can cite the work of Goorbergh, Genest, and Werker (2005) in option pricing.


**Pair copula construction**

The PCC is a very flexible construction, which allows the free specification of \( n(n-1)/2 \) bivariate copulas. This construction was proposed in the seminal paper by Joe (1996), and it has been discussed in detail, especially for applications in simulation and inference (Bedford & Cooke, 2001, 2002; Kurowicka & Cooke, 2006). The PCC is hierarchical by nature. The modelling scheme is based on the decomposition of a multivariate density into \( n(n-1)/2 \) bivariate copula densities, of which the first \( n-1 \) are dependency structures of unconditional bivariate distributions, and the rest are dependency structures of conditional bivariate distributions (Aas & Berg, 2011).

The PCC is usually represented in terms of density. The two main types of PCC that have been proposed in the literature are the C (canonical)-vines and D-vines. In the present paper we focus on the D-vine estimation, which according to Aas, Czado, Frigessi, and Bakken (2009) has the density as in formulation (8).

\[ f(x_0, \cdots, x_n) = \prod_{j=1}^{n-1} f(x_j) \prod_{j=1}^{n-2} \prod_{j \neq i}^{n} C_{ij}(x_i | x_{ij}, \cdots, x_{ij-1}, x_{ij+1}, \cdots, x_{ij+n-1}) \]  

(8)

In (8), \( x_1, \ldots, x_n \) are variables; \( f \) is the density function; \( C_{ij} \) is a bivariate copula density and the conditional distribution functions are computed, according to Joe (1996), by formulation (9).

\[ F(x | v) = \frac{\partial C_{xv | j} \{F(x | v_j), F(v_j | v_{-j})\}}{\partial v_j} \]  

(9)

In (9), \( C_{xv | j} \) is the dependency structure of the bivariate conditional distribution of \( x \) and \( v \) conditioned on \( v_{-j} \), where the vector \( v_{-j} \) is the vector \( v \) excluding the component \( v_j \). In order to make it possible to use the D-vine construction to represent a dependency structure through copulas, we must assume that the univariate margins are uniform in the interval [0, 1]. As an illustration, we present in formulation (10) a four-dimensional case, and its graphical representation in Fig. 1.

\[ C(u_1, u_2, u_3, u_4) = C_{12}(u_1, u_2) \cdot C_{34}(u_3, u_4) \cdot C_{13}(F(u_1 | u_2), F(u_3 | u_2)) \cdot C_{14}(F(u_1 | u_2), F(u_4 | u_2)) \cdot C_{23}(F(u_2 | u_1), F(u_3 | u_1)) \cdot C_{24}(F(u_2 | u_1), F(u_4 | u_1)) \cdot C_{31}(F(u_3 | u_1), F(u_1 | u_3)) \cdot C_{41}(F(u_4 | u_1), F(u_1 | u_4)) \]  

(10)
We collected daily yields for 1-, 2-, 3-, 5-, 7- and 10-years’ maturity Treasury bonds of the US government, from January 2, 1990 to April 12, 2012, totalling 5573 observations. These bonds were chosen because the US money market is traditionally considered by investors as the best source of risk-free assets. The choice of this period was due to the combination of the availability of data in the website of the US Department of Treasury and the need for collection of information over a length of time in order to avoid bandwidth biases. We excluded from the analysis those bonds that did not have negotiations during the whole period. Also, there is always a computational concern with the high-dimensionality of data, obligating the research to choose parsimoniously which variables to include in a model. Reinforcing the option for these bonds, it is worth mentioning that they are, in general, the most liquid.

Seeking to avoid issues relative to the non-stationary condition, we calculated the logarithmic differences (log-retuns) of the collected daily yields. We modelled the marginal of these log-returns through autoregressive moving average (ARMA \((m,n)\)) - generalised autoregressive conditional heteroscedastic (GARCH \((p,q)\)) models with student innovations, in order to consider the well-known conditional heteroscedastic heavy-tailed behaviour of the financial assets (Longin & Solnik, 2001). The estimated model is represented in formulations (11) to (13).

\[
r_{t} = \mu_{t} + \sum \phi_{s,m} r_{t-s-m} + \sum \eta_{t-n} e_{t-s-n} + \epsilon_{t, t}
\]

\[
e_{t, t} = h_{t} z_{t, t}, \quad z_{t, t} - t_{c}
\]
\[ h_{it} = \alpha_t + \sum \alpha_i \epsilon_{t-\phi}^2 + \sum \beta_i \epsilon_{t-\phi}^2, \]  
(13)

where \( r_{it} \) is the log-return of asset \( i \) in period \( t \); \( h_{it}^2 \) is the conditional variance of asset \( i \) in period \( t \); \( \mu_t, \phi, \theta_i, \alpha, \) and \( \beta \) are parameters; \( \epsilon_{t-\phi} \) is the innovation in the conditional mean of asset \( i \) in period \( t \); \( z_{it} \) represents the white noise of \( t \)-student distribution. The models were validated through the verification of serial correlation in the linear and squared standardised residuals through the \( Q \) statistic, represented for \( (14) \).

\[ Q = n(n+2) \sum_{k=1}^{n} \frac{\hat{\rho}_k^2}{n-k} \]  
(14)

In \( (14) \), \( n \) is the size of sample; \( \hat{\rho}_k \) is the autocorrelation of sample in lag \( k \); \( h \) is the number of lags being tested. The \( Q \) statistics, which test the null hypothesis of randomness of data, follow a chi-squared \( (\chi^2) \) distribution with \( h \) degrees of freedom.

After modelling the marginal, we estimated the PCC composed of the sector indexes. We standardised the residuals of the GARCH approach into pseudo-observations \( U_j = (U_{1j}, \ldots , U_{nj}) \) through the ranks as \( U_{ij} = R_{ij}/(n+1) \). We ordered the variables by decreasing order of the sum of the non-linear dependence, measured through Kendall’s tau, with the other variables. Subsequently, to choose the copula that best fit each bivariate pair of variables we employed the AIC criterion. A more detailed presentation of the copula families present in this selection is given in the appendix. The estimated parameters in the association measures presented some statistics for the daily log-returns of the U.S. Treasury bond.

Complementing this descriptive analysis, Table 1 presents some statistics for the daily log-returns of the U.S. Treasury bond yields. The results contained in Table 1 firstly indicate that the daily yields of the Treasury bonds had an expected value very close to zero, as pointed out by the central tendency measures. Moreover, the bonds presented a common evolution along the sample.

Results and discussion

This section is divided, for best comprehension, into two parts: i) marginal modelling, which exposes the descriptive characteristics of the studied data, as well as the results of the marginal models estimation; and ii) conditional dependence modelling, which presents the results for the estimated PCC, and also its implications for the Treasury bond market.

Marginal modelling

Initially we collect data for the daily yield for the US Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years of maturity. A study of the collected data for the daily yield for the US Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years of maturity (totalling 5573 observations) shows that the daily yields of all maturities presented a common evolution along the sample. This long term equilibrium is expected, since the bonds are linked by a mutual monetary policy which is managed through variables as the basic interest rate and the inflation. It is valid to emphasise that the expected yield rises with the maturity of its respective bond, as reflected by the well-known yield curves.

Another fact that emerges is the fall in the yield rates during periods of economic turbulence, as the 1993/1994 stagnation (observations 800–1200), the crisis in emerging markets in the decade of the 90s (observations 1500-2300), the terrorist attacks in 2001 (observations 3000-3500), the sub-prime and Euro-zone crisis (observations 4500 to the present). These falls in the yield are intrinsically linked with the attempt of the US government to promote the economy through an expansionist monetary policy with low interest rates, according to Miao, Wu, and Su (2013). It follows that this situation is very strong and continual in financial crises volatility and linked to the expectation of falling interest rates, where investors would prefer Treasury bond rather than short-term bond.

In order to avoid the non-stationary behaviour of the yield curves in level, we calculated their log-returns. The plots with the time series of the log-returns of the analysed Treasury bonds during the sample period are presented in Fig. 2. The plots in Fig. 2 reveal, again, the presence of a similar temporal evolution in the series. There were notorious volatility clusters during the turbulent periods previously mentioned, which coincide with the falls in the yields. This result corroborates the stylised fact of financial assets which attests that there is more volatility in falls than in rises. Moreover, the dispersion around the long term mean appears to be larger for the bonds with more maturity time during the calm periods, and for those with less maturity time during the periods of strong economic turbulence.

Complementing this descriptive analysis, Table 1 presents some statistics for the daily log-returns of the U.S. Treasury bond yields. The results contained in Table 1 firstly indicate that the daily yields of the Treasury bonds had an expected value very close to zero, as pointed out by the central tendency measures. Moreover, the bonds presented a great range (maximum—minimum) and dispersion (standard deviation) during the analysed period. This variability decreases with the time of maturity of the bond (same observations are made by Junker et al., 2006, when analysing 1-, 2-, 3-, 4- and 5-years of maturity from 1982 to 1991 and from 1992 to 2001 subsample periods), confirming the fact that the bonds with less time of maturity were more sensitive to the economic turbulences which occurred in the sample. Further, all series are leptokurtic and negative asymmetric.

These descriptive results confirm the well-known stylised facts about financial assets, previously cited in this paper. Thus, it is necessary to use flexible techniques in order to model both the marginal and the joint evolution of this kind.

1 We performed cointegration tests which statistically confirmed the long term equilibrium among all the analysed bonds.
2 We performed unit root tests in the series of the Treasury bond yields which showed as non-stationary in the level but stationary in the logarithmic difference.
3 We performed tests of normality which were rejected for all the series of the log-returns of the Treasury bond yields.
of variable. Regarding the marginal, Table 2 presents the estimated parameters, as well as the diagnostics of the ARMA-GARCH models utilised to model the studied log-returns.

The results contained in Table 2 clearly indicate that the daily log-returns of the yields were very persistent during the studied period, as one can perceive by the significance of the auto-regressive parameters. This influence of the lagged variations in the yields can be explained by the fact that the chosen period is very large and it contains some economic turbulence, which leads to a rise in the dependence on past information. Further, some bonds, especially those with longer time of maturity, presented a value for their unconditional mean significantly different from zero.

Regarding the conditional variance, all the log-returns of the US Treasury bond yields were significantly affected by the squared deviations from their expected value, as well as by the conditional variance from the last day of negotiation. Moreover, the estimated ARMA-GARCH models were validated through the Q statistic. The null hypothesis of no dependence on past information was not rejected for any of the bonds, both for the linear standardised residuals as for their quadratic form.

Complementing this, Fig. 3 exposes the daily conditional volatilities of the log-returns of the US Treasury bond yields in the sample period obtained through the ARMA-GARCH models. The plots visually confirm the previous results that infer a presence of volatility clusters in the cited turbulent periods. Again, the peak of the dispersion occurred during the sub-prime and Euro-zone crisis. Further, the level of the

![Figure 2](image)

**Figure 2** Daily log-returns of the yield of US Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years of maturity from January 2, 1990 to April 12, 2012, totalling 5573 observations.

<table>
<thead>
<tr>
<th>Years to maturity</th>
<th>One</th>
<th>Two</th>
<th>Three</th>
<th>Five</th>
<th>Seven</th>
<th>Ten</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>−0.2877</td>
<td>−0.3514</td>
<td>−0.3102</td>
<td>−0.2614</td>
<td>−0.2241</td>
<td>−0.1850</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.2962</td>
<td>0.3185</td>
<td>0.2097</td>
<td>0.1323</td>
<td>0.1169</td>
<td>0.0892</td>
</tr>
<tr>
<td>Mean</td>
<td>−0.0006</td>
<td>−0.0006</td>
<td>−0.0005</td>
<td>−0.0004</td>
<td>−0.0003</td>
<td>−0.0002</td>
</tr>
<tr>
<td>Median</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>St. deviation</td>
<td>0.0299</td>
<td>0.0310</td>
<td>0.0277</td>
<td>0.0217</td>
<td>0.0180</td>
<td>0.0149</td>
</tr>
<tr>
<td>Skewness</td>
<td>−0.1854</td>
<td>−0.1208</td>
<td>−0.2611</td>
<td>−0.3307</td>
<td>−0.2120</td>
<td>−0.2702</td>
</tr>
</tbody>
</table>

4 The significance level of 5% was chosen.
Table 2  Estimated parameters* and diagnostics** of the linear and squared residuals of the estimated ARMA-GARCH models* for the daily log-returns of the yield of U.S. Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years of maturity from January 2, 1990 to April 12, 2012, totalling 5573 observations.

<table>
<thead>
<tr>
<th>Years to maturity</th>
<th>One</th>
<th>Two</th>
<th>Three</th>
<th>Five</th>
<th>Seven</th>
<th>Ten</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ</td>
<td>0.0009 (0.9440)</td>
<td>−0.0002 (0.1877)</td>
<td>−0.0003 (0.0071)</td>
<td>−0.0003 (0.0727)</td>
<td>−0.0003 (0.0162)</td>
<td>−0.0003 (0.0012)</td>
</tr>
<tr>
<td>ϕ_1</td>
<td>0.0384 (0.0040)</td>
<td>0.0382 (0.0084)</td>
<td>0.0435 (0.0004)</td>
<td>0.0417 (0.0003)</td>
<td>−0.0427 (0.0010)</td>
<td>−0.0319 (0.0017)</td>
</tr>
<tr>
<td>ϕ_2</td>
<td>−0.0256 (0.0407)</td>
<td>−0.0326 (0.0212)</td>
<td>−0.0422 (0.0193)</td>
<td>−0.0359 (0.0059)</td>
<td>−0.0318 (0.0028)</td>
<td>0.0006 (0.0230)</td>
</tr>
<tr>
<td>α</td>
<td>0.0615 (0.0000)</td>
<td>0.0577 (0.0000)</td>
<td>0.0568 (0.0000)</td>
<td>0.0499 (0.0000)</td>
<td>0.0484 (0.0000)</td>
<td>0.0176 (0.0057)</td>
</tr>
<tr>
<td>ω</td>
<td>0.0000 (0.4902)</td>
<td>0.0000 0.3902</td>
<td>0.0000 (0.3312)</td>
<td>0.0000 (0.2474)</td>
<td>0.0000 (0.2155)</td>
<td>0.0000 (0.1529)</td>
</tr>
<tr>
<td>ϕ_3</td>
<td>0.0240 (0.0169)</td>
<td>0.0308 (0.0169)</td>
<td>0.0249 (0.0114)</td>
<td>0.0271 (0.0056)</td>
<td>−0.0141 (0.034)</td>
<td></td>
</tr>
<tr>
<td>ϕ_4</td>
<td>−0.0070 (0.0033)</td>
<td>0.0121 (0.0074)</td>
<td>0.0245 (0.0113)</td>
<td>0.0267 (0.0132)</td>
<td>0.0267 (0.0132)</td>
<td>0.0267 (0.0132)</td>
</tr>
<tr>
<td>ϕ_5</td>
<td>0.9375 (0.0000)</td>
<td>0.9413 (0.0000)</td>
<td>0.9422 (0.0000)</td>
<td>0.9486 (0.0000)</td>
<td>0.9491 (0.0000)</td>
<td>0.9505 (0.0000)</td>
</tr>
<tr>
<td>ϕ_6</td>
<td>0.0615 (0.0000)</td>
<td>0.0577 (0.0000)</td>
<td>0.0568 (0.0000)</td>
<td>0.0504 (0.0000)</td>
<td>0.0499 (0.0000)</td>
<td>0.0484 (0.0000)</td>
</tr>
<tr>
<td>ϕ_7</td>
<td>0.9375 (0.0000)</td>
<td>0.9413 (0.0000)</td>
<td>0.9422 (0.0000)</td>
<td>0.9486 (0.0000)</td>
<td>0.9491 (0.0000)</td>
<td>0.9505 (0.0000)</td>
</tr>
<tr>
<td>ϕ_8</td>
<td>0.0615 (0.0000)</td>
<td>0.0577 (0.0000)</td>
<td>0.0568 (0.0000)</td>
<td>0.0504 (0.0000)</td>
<td>0.0499 (0.0000)</td>
<td>0.0484 (0.0000)</td>
</tr>
<tr>
<td>ϕ_9</td>
<td>0.0615 (0.0000)</td>
<td>0.0577 (0.0000)</td>
<td>0.0568 (0.0000)</td>
<td>0.0504 (0.0000)</td>
<td>0.0499 (0.0000)</td>
<td>0.0484 (0.0000)</td>
</tr>
<tr>
<td>ϕ_10</td>
<td>0.0000 (0.4902)</td>
<td>0.0000 0.3902</td>
<td>0.0000 (0.3312)</td>
<td>0.0000 (0.2474)</td>
<td>0.0000 (0.2155)</td>
<td>0.0000 (0.1529)</td>
</tr>
<tr>
<td>Shape</td>
<td>4.7517 (0.0000)</td>
<td>5.6357 (0.0000)</td>
<td>6.3832 (0.0000)</td>
<td>6.5653 (0.0000)</td>
<td>7.2488 (0.0000)</td>
<td>7.2429 (0.0000)</td>
</tr>
<tr>
<td>Q(10) Linear</td>
<td>14.1011 (0.1685)</td>
<td>2.9290 (0.9831)</td>
<td>5.1855 (0.9767)</td>
<td>2.6153 (0.9891)</td>
<td>2.4923 (0.9510)</td>
<td>2.9776 (0.9820)</td>
</tr>
<tr>
<td>Q(15) Linear</td>
<td>17.4345 (0.2870)</td>
<td>12.4981 (0.6410)</td>
<td>12.9482 (0.6063)</td>
<td>15.9312 (0.3866)</td>
<td>12.5431 (0.6375)</td>
<td>13.8184 (0.5394)</td>
</tr>
<tr>
<td>Q(20) Linear</td>
<td>19.3343 (0.4989)</td>
<td>19.5223 (0.4882)</td>
<td>17.4312 (0.6248)</td>
<td>20.8033 (0.4088)</td>
<td>18.4712 (0.5564)</td>
<td>18.2255 (0.5726)</td>
</tr>
<tr>
<td>Q(10) Squared</td>
<td>14.1213 (0.1656)</td>
<td>8.1576 (0.6135)</td>
<td>8.7964 (0.5516)</td>
<td>8.3882 (0.5910)</td>
<td>6.4393 (0.7771)</td>
<td>5.4891 (0.8562)</td>
</tr>
<tr>
<td>Q(15) Squared</td>
<td>17.4109 (0.2950)</td>
<td>11.6987 (0.7017)</td>
<td>12.0212 (0.6775)</td>
<td>12.3082 (0.6556)</td>
<td>10.825 (0.7649)</td>
<td>(11.543) (0.7095)</td>
</tr>
<tr>
<td>Q(20) Squared</td>
<td>19.3122 (0.5018)</td>
<td>16.5712 (0.6807)</td>
<td>15.4634 (0.7493)</td>
<td>15.5954 (0.7414)</td>
<td>15.3625 (0.7573)</td>
<td>15.0922 (0.7711)</td>
</tr>
</tbody>
</table>

*Parameters are defined in (11) and (13). Shape is the number of degrees of freedom of the student’s t conditional distribution.

**Q(k) is the statistic for k lags; p-values are in parenthesis.

We chose to limit the number of auto-regressive parameters to 10 for computational and parsimony issues. However, there was no need for more lagged parameters, as explicated by the Q statistics.
variability of the bonds with less time of maturity was higher than that of the bonds with more time of maturity.

**Conditional dependence modelling**

In this step we model the dependence structure of the Treasury bonds isolating the effect of the marginal, which were modelled through the ARMA-GARCH models. Initially, we present in Fig. 4 the scatter plot of the residuals of the marginal models. The scatter plot of Fig. 4 indicates that all the bivariate associations between the daily variations of the yields are strongly positive (Junker et al., 2006 use normal copula yields and confirm this same correlation). This characteristic of dependence reflects, in certain degree, the long term association of the yield curve of the bonds. Moreover, the plots point out that there are associations in the extreme values (tails) of the presented relationships. This behaviour is a vestige of the need for joint distribution that considers this probability in the tails.

Subsequently, we calculated the matrix of dependence for the daily log-returns of Treasury bond yields, through the Kendall’s tau measure, aiming to select their order in the estimation of the PCC. The adopted criterion was the absolute sum of the dependence between each index with all the others. The results are presented in Table 3. The results in Table 3 reinforce the presence of great dependence between the Treasury bonds. With the exception of the pair 1 year/10 years, all the relationships had magnitude of the non-linear dependence over 0.5. The mean magnitude of the associations was 0.69, a very large value.

Regarding the order, the bond with most dependence with the others was the 5-years, followed by 3-years, 7-years, 10-years, 2-years and 1-year of maturity. With this order we estimate, through ML, a PCC for the log-returns of the Treasury bond yields in the sample period. The results of this estimation, as well as the dependence measures associated with the parameters of the pair copulas, are presented in Table 4.

The results in Table 4 initially indicate that there is an absolute predominance of the Student’s t copula in the bivariate relationships which compose the dependence structure of the US government Treasury bonds. This result corroborates that of previous research, such as that performed by Marshal and Zeevi (2002) and Diks et al. (2014), which have shown that the fit of this copula family is generally superior to that of other copulas for financial data. Based on the selected families of the PCC estimation, it is noteworthy that these copulas assign, in certain degree, more importance to the tails of the joint probability distribution than the Gaussian one. This suggests that there is more dependence among the sectors in extreme events than in the normally expected events.

Table 4 presents the dependence measures, converted through the estimated copulas of each bivariate relationship. Firstly, all the measures (lower tail, upper tail, and tau) exhibited a trend for decreasing behaviour in the direction of the initial levels of the vine to the final ones, which was
expected as this is the nature of this hierarchical construction. However, some relationships in the last levels of the vines exhibited large association, as for example, the association between the bonds of 3-years and 2-years of maturity. The separations of dependence measures in terms of maturity (attributed to Lee et al., 2011) denote the presence of different dealing participants: active dealers tend to short-term maturities bonds and passive dealers tend to long-term maturities.

Regarding the magnitude of the dependence, the tail measures obtained relevant values in most of the bivariate relationships, except those in the last levels of the vine where even the absolute dependence (tau) was very low. The tail dependences were very similar to the absolute one almost in all cases. The dependences in the lower and upper tails were equal, reflecting the fact that the Student’s t copula is elliptical. It is noteworthy that the relationship between the bonds with 3-years and 10-years in the estimated PCC obtained negative sign, emphasising the differences that are verified in the dependences between two variables when one isolates the effect of other variables.

Further, the estimated PCC rejected the null hypothesis of the Clark test, which states that there is significant distinction in the fit of the utilised D-vine approach and the C-vine construction, emphasising the advantages in choosing the D-vine construction. Regarding the domain of the dependence, the 7-year bond presented the greatest mean for the

**Figure 4** Scatter plot of the estimated residuals of the ARMA-GARCH models for the daily log-returns of the yield of US Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years of maturity from January 2, 1990 to April 12, 2012, totalling 5573 observations.

**Table 3** Kendall’s tau* dependence matrix of the log-returns of the yield of the U.S. Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years of maturity from January 2, 1990 to April 12, 2012, totalling 5573 observations.

<table>
<thead>
<tr>
<th>Years</th>
<th>One</th>
<th>Two</th>
<th>Three</th>
<th>Five</th>
<th>Seven</th>
<th>Ten</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>1.0000</td>
<td>0.6392</td>
<td>0.5980</td>
<td>0.5624</td>
<td>0.5199</td>
<td>0.4912</td>
</tr>
<tr>
<td>Two</td>
<td>0.6392</td>
<td>1.0000</td>
<td>0.7071</td>
<td>0.7294</td>
<td>0.6654</td>
<td>0.6214</td>
</tr>
<tr>
<td>Three</td>
<td>0.5980</td>
<td>0.7971</td>
<td>1.0000</td>
<td>0.8149</td>
<td>0.7471</td>
<td>0.6976</td>
</tr>
<tr>
<td>Five</td>
<td>0.5624</td>
<td>0.7294</td>
<td>0.8149</td>
<td>1.0000</td>
<td>0.8539</td>
<td>0.7993</td>
</tr>
<tr>
<td>Seven</td>
<td>0.5199</td>
<td>0.6654</td>
<td>0.7471</td>
<td>0.8539</td>
<td>1.0000</td>
<td>0.8680</td>
</tr>
<tr>
<td>Ten</td>
<td>0.4912</td>
<td>0.6214</td>
<td>0.6976</td>
<td>0.7993</td>
<td>0.8680</td>
<td>1.0000</td>
</tr>
<tr>
<td>Sum</td>
<td>3.8107</td>
<td>4.4523</td>
<td>4.6547</td>
<td>4.7599</td>
<td>4.6543</td>
<td>4.4775</td>
</tr>
</tbody>
</table>

*The Kendall’s tau measure was chosen because it can identify non-linear dependence, unlike the traditional linear correlation.
Table 4  Pair copula constructions* of the daily log-returns of the yield of the U.S. Treasury bonds with 1-, 2-, 3-, 5-, 7- and 10-years** of maturity from January 2, 1990 to April 12, 2012, totalling 5573 observations.

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Parameters</th>
<th>Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copula</td>
<td>Family</td>
<td>First</td>
</tr>
<tr>
<td>C&lt;sub&gt;5.3&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.9540</td>
</tr>
<tr>
<td>C&lt;sub&gt;5.7&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.9169</td>
</tr>
<tr>
<td>C&lt;sub&gt;7.10&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.9751</td>
</tr>
<tr>
<td>C&lt;sub&gt;10.2&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.8203</td>
</tr>
<tr>
<td>C&lt;sub&gt;2.1&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.8699</td>
</tr>
<tr>
<td>C&lt;sub&gt;5.7&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.7992</td>
</tr>
<tr>
<td>C&lt;sub&gt;3.10q&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>-0.1639</td>
</tr>
<tr>
<td>C&lt;sub&gt;7.3p&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.4958</td>
</tr>
<tr>
<td>C&lt;sub&gt;10.4p&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.0305</td>
</tr>
<tr>
<td>C&lt;sub&gt;5.10q.7&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.1120</td>
</tr>
<tr>
<td>C&lt;sub&gt;3.2p.10&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.7888</td>
</tr>
<tr>
<td>C&lt;sub&gt;7.4p.2&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.0196</td>
</tr>
<tr>
<td>C&lt;sub&gt;5.2p.7.10&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.1715</td>
</tr>
<tr>
<td>C&lt;sub&gt;3.4p.10.2&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.0922</td>
</tr>
<tr>
<td>C&lt;sub&gt;5.4p.7.10.2&lt;/sub&gt;</td>
<td>Student’s t</td>
<td>0.0328</td>
</tr>
<tr>
<td><strong>Clark test</strong></td>
<td>518.0</td>
<td>p-value</td>
</tr>
</tbody>
</table>

*Selected families and their estimated parameters. These parameters were converted in the lower tail, upper tail and Kendall’s tau dependence measures.

**To facilitate the interpretation of the relationships, we use the number of years of maturity of each bond.

tau and tail measures, if considered in relation to the other bonds. The 5-year, which had the largest association with the others, lost dependence after the isolation of indirect effect. This can be explained by the liquidity in the negotiation of these bonds. Chordia, Sarkar, and Subrahmanyam (2005), in their in-depth study of the relationship of bonds with liquidity, find the association between monetary expansions and increased liquidity, in which government bond sector plays an important role in forecasting bond market liquidity.

In a general form, these results highlight the importance of risk management in terms of bonds diversification. This is because such concentration of joint probability in the tails, in particular for lower values, indicates that it can be difficult to minimise the risk of a portfolio based on investment allocation in these bonds, especially in times of negative innovations, such as a crisis, which is when active managers most need to protect their investments. Junker et al. (2006) seek to focus on this approach; however, they do not delve in depth on the relationship between Treasury bond maturities since they work is more restricted to the comparison between families of copulas.

**Conclusion**

This paper aimed to estimate the dependence structure between Treasury bonds through a PCC. To that effect, we used data from the US government Treasury bonds for 1-, 2-, 3-, 5-, 7- and 10-years of maturity. Initially we verified that the daily yields presented a common evolution along the sample. This long term equilibrium reflects the influence of the monetary policy (see work of Chordia et al., 2005). In that sense, there were falls in the yield rates during periods of economic turbulence, which were intrinsically linked with the attempt of the government to promote the economy through an expansionist monetary policy with low interest rates.

Further, we realised that the variability of the yields decreases with the time taken for maturity of the bond, confirming the fact that the bonds with less maturity time were more sensitive to economic turbulences. Moreover, the yields presented strong dependence with past values, as emphasised by the results of the marginal models. With the residuals of the marginal models, which are isolated from the marginal distribution, we found that all the bivariate associations between the daily variations of the yields were strongly positive and with associations in the tails.

Subsequently, with the results of the estimated PCC, we could verify that there is an absolute predominance of the Student’s t copula in the relationships between the bonds. Differing from Junker et al. (2006), who use normal copula yields, these copulas assign dependence in the extreme values, relevant in scenarios of crises. Regarding dependence, the tail measures obtained relevant values in most of the relationships, and were similar to the absolute one in practically all cases. In terms of domain, the 7-year bonds presented the greater mean for the tau and tail measures, when considered in relation with the other bonds. The 5-year bonds, which had the largest association with the others in the previous step, lost dependence after the isolation of indirect effect. This can be explained by the liquidity in the negotiation of these bonds, especially the “flight-to-quality” of passive dealers in stable long-term maturities. This isolation also reduced significantly the magnitude of some relationships and even changed the sign of one association.

These results highlight the importance of risk management in terms of bonds diversification. This is because such
concentration of joint probability in the tails, in particular for lower values, indicates that it can be difficult to minimise the risk of a portfolio based on investment allocation in these bonds, especially in times of negative innovations, such as a crisis, which is when managers most need to protect their investments. Further, the PCC is less restrictive on degree of dependence than Archimedean structure defended by Lee et al. (2011) and enables best performance of diversification.

For future research we suggest the estimation of PCC in order to determine the dependence structure of commodities and other kinds of financial assets. Regarding Treasury bonds, we recommend the comparison between the association of their dependence in emerging and developed markets, seeking to identify possible differences in the monetary policy, and risk implications in Treasury bond portfolios.

Appendix
In this appendix we present the families of copulas which were candidates to fit the bivariate relationships between the log-returns of the US Treasury bonds. The families utilised were elliptical (Normal and Student’s t) and Archimedean (Clayton, Gumbel, Frank, Joe, BB1, BB7 and BB8).

The elliptical families are characterised by the symmetry. Let $\rho$ be the bivariate linear correlation. The Normal (or Gaussian) and Student’s t copulas are defined, respectively, in (A1) and (A2).

$$C^{\text{Gau}}(u, v) = \Phi_s(\Phi^{-1}(u), \Phi^{-1}(v)). \quad (A1)$$

$$C^{\text{Std}}(u, v) = t_n(u, t_n(v)). \quad (A2)$$

In (A1) and (A2), $\Phi^{-1}$ is the inverse of the standard univariate normal distribution function; $t_n$ is the inverse of the univariate Student’s t distribution function with $\nu$ degrees of freedom.

The Archimedean copulas may be constructed using a function called generator. Let $\alpha$ and $\beta$ be parameters. The Clayton, Gumbel, Frank, Joe, BB1, BB6, BB7 and BB8 copulas are represented, respectively, by formulations (A3) to (A10).

$$C^{\text{Clay}}(u, v) = \max \left[ u^{-\alpha} + v^{-\alpha} - 1, 0 \right], \alpha \in (-1, 0) \cup (0, +\infty) \quad (A3)$$

$$C^{\text{Gum}}(u, v) = \exp \left[ -\left( -\ln u \right)^\alpha - \left( -\ln v \right)^\alpha \right], \alpha \in (1, +\infty) \quad (A4)$$

$$C^{\text{Frank}}(u, v) = -\frac{1}{\alpha} \ln \left( 1 + \frac{\exp(-\alpha u) - 1)(\exp(-\alpha v) - 1)}{\exp(-\alpha)} \right), \alpha \in (-\infty, 0) \cup (0, +\infty). \quad (A5)$$

$$C^{\text{Joe}}(u, v) = 1 - \left[ (1 - u)^\alpha + (1 - v)^\alpha - (1 - u)^\alpha(1 - v)^\alpha \right]^{1/\alpha}, \alpha \in (1, +\infty) \quad (A6)$$

$$C^{\text{BB1}}(u, v) = \left[ 1 + \left( u^{-\alpha} - 1 \right)^\beta + \left( v^{-\alpha} - 1 \right)^\beta \right]^{\frac{1}{\beta}}, \alpha > 0, \beta \geq 1 \quad (A7)$$

$$C^{\text{BB8}}(u, v) = 1 - \exp \left[ -\left( -\log(1 - (1 - u)^\alpha) \right)^\beta + \left( -\log(1 - (1 - v)^\alpha) \right)^\beta \right], \alpha \geq 1, \beta \geq 1. \quad (A8)$$

$$C^{\text{BB8}} = 1 - \left\{ 1 - \left[ (1 - (1 - u)^\alpha)^\beta + (1 - (1 - v)^\alpha)^\beta - 1 \right]^{-1} \right\}^{1/\beta}, \alpha \geq 0, \beta \geq 1. \quad (A9)$$

$$C^{\text{BB8}} = \frac{1}{\beta} \left[ 1 - \left[ (1 - (1 - \beta)^\alpha)^\beta + (1 - (1 - \beta)^\alpha)^\beta - 1 \right]^{-1} \right], \alpha \geq 1, 0 \leq \beta \leq 1. \quad (A10)$$

Further, we also utilised rotated versions of the presented copulas, with the exception for the Normal and Student’s t families. When rotating the copulas by 180 degrees, one obtains the corresponding survival copulas, while rotation by 90 and 270 degrees allows the modelling of negative dependence which is not possible with the standard non-rotated versions.

References


