Interference and Outage Probability Analysis for Massive MIMO Downlink with MF Precoding

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***Abstract*—This letter analyzes the user-interference and outage probability for single-cell multi-user massive multi-input-multi-output (MIMO) systems with matched-filter (MF) precoding. Existing performance studies on massive MIMO systems have focused on the sum-rate by deriving the asymptotic deterministic equivalence. In this work, we treat the user-interference as random, and derive a tight closed-form approximation for the distribution of the interference power. This enables the analysis of the outage probability. The derived results are shown to have accurate match with the simulation.**

***Index Terms*—Interference modeling, massive MIMO, MF precoding, multi-user MIMO, outage probability.**

I. Introduction

IN RECENT years, massive MIMO systems, where the base station (BS) is equipped with hundreds antennas, emerge as one key concept for the next generation wireless systems. Massive MIMO can achieve all merits of conventional MIMO systems with a much greater scale [1]. In addition, as the number of BS antennas increases to infinity, the intra-cell interference and the noise can be averaged out due to the law of large numbers [2]. There have been many results on different aspects of massive MIMO [3]–[6].

As to the performance analysis of massive MIMO, existing work focused on the sum-rate [7]–[9], where the asymptotic deterministic equivalence of the signal-to-interference-plus-noise ratio (SINR) was derived for different massive MIMO scenarios. However, there are little results on other important performance measure such as the outage probability. We found three papers [10]–[12] related to the outage analysis. In [10], the secrecy outage capacity of a massive MIMO relaying system was studied, where the SINR at the eavesdropper is approximated based on channel hardening and an approximate SINR distribution was derived. In [11], the secrecy outage probability of multi-cell massive MIMO systems was analyzed, where the SINR at the eavesdropper was shown to be equivalent to the SINR of a multiple-branch minimum-mean-square-error diversity combiner. For a point-to-point massive MIMO channel, [12] proved that the mutual information approaches Gaussian as the dimension increases, and the outage probability was approximated using the Q-function.

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In this work, we analyze the outage probability of the downlink of a single-cell multi-user massive MIMO system with matched-filter (MF) precoding. Other than the theoretical sum-rate, the outage probability is also important in evaluating the user experience. For massive MIMO, although the sum-rate can increase with more users, we will show that an excessively large number of users can result in a low SINR and a large outage probability, which is undesirable for carriers and customers. This motivates the outage probability analysis in this work. However, existing analysis for massive MIMO relies on asymptotic deterministic equivalence to derive the sum-rate, which cannot be used for outage probability. Due to the large difference in system model, the outage probability analysis in [10]–[12] do no apply for multi-user massive MIMO downlink either. In this work, we propose a new analytical method to study the outage probability. Compared to the deterministic equivalence analysis, it preserves the randomness in the SINR to enable the outage probability calculations.

Our novelty and contribution can be summarized as follows.

- For the multi-user massive MIMO downlink with MF precoding, we conduct a refined analysis on the distribution of the user-interference power, and derive its asymptotic probability density function (pdf) in closed-form.
- Based on the interference analysis and calculations on the variances of the signal power and interference power, an approximation on the outage probability is derived, which shows the outage probability behaviour with respect to different network parameters.
- Simulations show that our analytical results are accurate. Besides, for a massive MIMO system with a large but finite number of antennas, the outage probability increases rapidly to 1 as the number of users increases. Also, for given numbers of BS antennas and users, the outage probability does not decrease to zero as the total transmit power increases due to user-interference.

The rest of the paper is organized as follows. In Section II, we elaborate the multi-user massive MIMO system model. In Section III, the pdf of the user-interference is derived and its properties are discussed. An analytical outage probability expression is derived in Section IV. Simulation results are given in Section V and we draw conclusions in Section VI.

II. Multi-User Massive MIMO System Model

We consider a single-cell multi-user massive MIMO system which has a BS and K single-antenna users. The BS is equipped with M antennas where M is very large ($M \gg 1$), e.g., a few hundreds [1], [2]. Rayleigh flat-fading channels are considered.

Let $\mathbf{h}_k$ be the $1 \times M$ channel vector from the BS antennas to the kth user. Entries of $\mathbf{h}_1, \cdots, \mathbf{h}_K$ are independent and identically distributed (i.i.d.) each following $\mathcal{CN}(0,1)$, the circularly symmetric complex Gaussian distribution with zero-mean and unit-variance. In addition, perfect CSI at the BS is assumed.

Let $x_1, \cdots, x_K$ be the independent data symbols intended for the K users under the normalization: $\mathbb{E}(x_k^2) = 1$, where $\mathbb{E}$ is the expectation operator. MF precoding is considered, where the symbol of each user is pre-coded by the Hermitian of its channel vector. MF precoding is a popular scheme for massive MIMO due to its low computational complexity, robustness, and high asymptotic performance. With MF precoding, the transmitted signal vector from the BS to all users is

$$\mathbf{s} = \sqrt{\frac{P_t}{KM}} \sum_{k=1}^{K} \mathbf{h}_k^{\mathrm{H}} x_k,$$

where P_t is the average total transmit power of the BS and $(\cdot)^{\mathrm{H}}$ denotes the matrix Hermitian.

The received signal at the kth user is given by

$$y_k = \sqrt{\frac{P_t}{KM}} \mathbf{h}_k \mathbf{h}_k^{\mathrm{H}} x_k + \sqrt{\frac{P_t}{KM}} \sum_{j=1, j\neq k}^{K} \mathbf{h}_k \mathbf{h}_j^{\mathrm{H}} x_j + n_k, \quad (1)$$

where n_k is the additive Gaussian noise with zero-mean and unit-variance. All noises are assumed to be independent with unit-power. The SINR of the kth user can be calculated as

$$\mathrm{SINR}_k = \frac{\frac{P_t}{KM} |\mathbf{h}_k \mathbf{h}_k^{\mathrm{H}}|^2}{1 + \frac{P_t}{KM} \sum_{j=1, j\neq k}^{K} |\mathbf{h}_k \mathbf{h}_j^{\mathrm{H}}|^2}. \quad (2)$$

To understand the performance of the massive MIMO system, we analyze the statistical properties of SINR_k. Especially, the statistical properties of the interference term in the denominator of (2) are crucial.

III. Analysis on the Interference Power

In this section, we analyze the user-interference. Instead of using asymptotic deterministic equivalence to find the average interference power, we study its random behaviour and derive a closed-form approximation of its pdf. Discussions on the properties of the pdf are also provided.

To help the presentation, we use Y_k to denote the power of the interference experienced by User k, i.e.,

$$Y_k \triangleq \frac{1}{M} \sum_{j=1, j\neq k}^{K} |\mathbf{h}_k \mathbf{h}_j^{\mathrm{H}}|^2.$$

The following proposition is proved.

Proposition 1: Define

$$\eta = \frac{K-1}{\sqrt{M} + K - 2}. \quad (3)$$

When $M \gg 1$, the pdf of Y_k has the following approximation:

$$f_{Y_k}(y) = (1-\eta) \sum_{i=0}^{\infty} \eta^i \phi\left(y; K+i-1, 1 - \frac{1}{\sqrt{M}}\right), \quad (4)$$

where $\phi(y;\alpha,\beta) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}(\alpha-1)!}, y > 0$ is the pdf of Gamma distribution with shape parameter α and scale β.

Proof: See the appendix. ■

Next, we discuss the properties of the pdf for the interference power. It can be seen from (4) that the interference power has a mixture distribution of infinite Gamma random variables with the same scale parameter $1 - 1/\sqrt{M}$ but different shape parameters. Also, the distribution is independent of the user index k.

From Proposition 1, the asymptotic deterministic equivalences of the SINR, and thus the system sum-rate can be derived. With (4), by straightforward calculations, we have $\mathbb{E}(Y_k) = K - 1$. Denote

$$X_k \triangleq \frac{1}{M} |\mathbf{h}_k \mathbf{h}_k^H|.$$

Since entries of $\mathbf{h}_k$ are i.i.d. following $\mathcal{CN}(0,1)$, X_k has the pdf of Gamma distribution $\phi(y; M, 1/M)$. Thus, $\mathbb{E}(X_k^2) = 1 + 1/M$. Under the law of large numbers and by approximating the interference power with its expectation, the asymptotic deterministic equivalence of the SINR can be obtained as

$$\mathrm{SINR}_{k,asym} = \frac{M}{K} \frac{P_t(1 + \frac{1}{M})}{1 + P_t \frac{K-1}{K}}. \quad (5)$$

When $K, M \to \infty$ but with fixed ratio, we have

$$\mathrm{SINR}_{k,asym} \to \frac{M}{K} \frac{P_t}{1 + P_t}, \quad (6)$$

which is the same as the SINR result derived in [7]. The asymptotic sum-rate is given by $R_{\mathrm{asym}} = K \log(1 + \mathrm{SINR}_{k,asym})$. The result in (5) has a tighter match with the simulated average SINR than (6), especially for small K and finite M. The figure is not shown due to the space limit.

The interference power pdf in (4) is in an infinite summation form. In reality, we can only evaluate it with finite terms. An approximation with the first L terms is as follows:

$$f_{Y_k,L}(y) = \frac{1-\eta}{1-\eta^L} \sum_{i=0}^{L-1} \eta^i \phi\left(y; K+i-1, 1 - \frac{1}{\sqrt{M}}\right). \quad (7)$$

The coefficient $(1-\eta^L)^{-1}$ is to guarantee $\int_0^{\infty} f_{Y_k,L} \mathrm{d}y = 1$. When $L = 1$, we get

$$f_{Y_k,1}(y) = \phi\left(y; K-1, 1 - \frac{1}{\sqrt{M}}\right). \quad (8)$$

This $L = 1$ approximation can also be obtained by assuming that the $K - 1$ terms in Y_k are independent to each other.

But notice that with the same scale, Gamma distribution with a larger shape parameter has a large tail. The approximations in (7) and (8) ignore large tail terms. They can be loose on the distribution tail and are inappropriate in the outage probability derivation. In what follows, we derived an closed-form formula for the pdf, which enables us to analyze the outage probability accurately.

Corollary 1: The pdf of Y_k can be rewritten into the following closed-form:

$$f_{Y_k}(y) = \frac{\sqrt{M}}{\sqrt{M}+K-2}\eta^{-(K-2)}\left[e^{-\frac{\sqrt{M}}{\sqrt{M}+K-2}y} - e^{-\frac{\sqrt{M}}{\sqrt{M}-1}y}\sum_{n=0}^{K-3}\left(\frac{\sqrt{M}}{\sqrt{M}-1}\eta\right)^n\frac{y^n}{n!}\right]. \quad (9)$$

Proof: Notice that

$$\sum_{i=0}^{\infty}\eta^i\phi\left(y;K+i-1,1-\frac{1}{\sqrt{M}}\right) = \frac{\sqrt{M}}{\sqrt{M}-1}\eta^{-(K-2)}e^{-\frac{\sqrt{M}}{\sqrt{M}-1}y}\left(\sum_{n=0}^{\infty}-\sum_{n=0}^{K-3}\right) \times\left(\frac{\sqrt{M}}{\sqrt{M}-1}\eta\right)^n\frac{y^n}{n!}.$$

By Taylor series for exponential function and straightforward calculations, we can obtain (9). ■

IV. Outage Probability Analysis

In this section, we derive the outage probability based on the interference power pdf provided in (9). Let

$$P_u \triangleq P_t/K,$$

which is the transmit power per user. From (2), the SINR of User k can be written as $\mathrm{SINR}_k = P_u M\cdot X_k^2/(1+P_uY_k)$. We first study the variances of X_k^2 and $1+P_uY_k$ respectively. With straightforward calculations, we have

$$\mathrm{Var}(X_k^2) = 4/M + \mathcal{O}(1/M^2).$$

For the interference term, by using the pdf in (4), we can show

$$\mathrm{Var}(Y_k) = K-1+(K-1)(K-2)/M.$$

Thus, the variance of $1+P_uY_k$ is given by

$$\mathrm{Var}(1+P_uY_k) = P_u^2\left[K-1+\frac{(K-1)(K-2)}{M}\right] > \frac{P_t^2(K-1)}{K^2}.$$

When $M\to\infty$, the variance of the desired signal power X_k^2 decreases to 0, meaning that the signal power becomes deterministic. However, this is not the case for the interference power, whose variance is not negligible for reasonable K and P_t, and is significantly larger than the variance of the signal power when $P_t^2M \gg K$. Thus for tractable analysis, we treat X_k^2 as deterministic and approximate it by its average. This is the same as using asymptotic deterministic equivalence for $M\to\infty$. But different to existing work, we keep Y_k as a random variable in the outage probability analysis below.

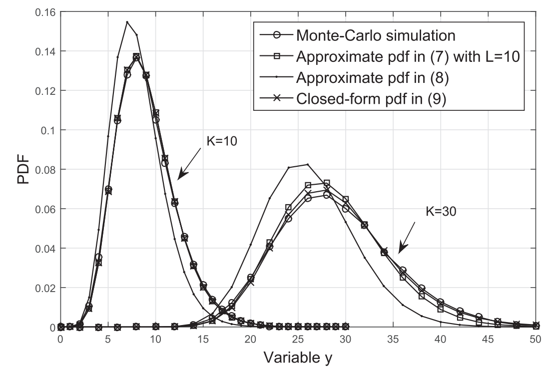

Fig. 1. PDF of Y_1 where $M = 100$.

Let γ_{th} be the SINR threshold. The outage probability of User k can thus be approximated as follows.

$$P_{\mathrm{out}} = \mathbb{P}\left(P_uM\frac{X_k^2}{1+P_uY_k}<\gamma_{th}\right) \approx \mathbb{P}\left(P_uM\frac{1+\frac{1}{M}}{1+P_uY_k}<\gamma_{th}\right) = \begin{cases} 1 & \text{if } \gamma_{th}\geq MP_u \\ \mathbb{P}\left(Y_k>\frac{M+1}{\gamma_{th}}-\frac{1}{P_u}\right) & \text{otherwise}\end{cases}.$$

When $\gamma_{th}\leq MP_u$, from (9), we have the outage probability results as shown in (10), where $\Gamma(s,x)\triangleq\int_x^\infty t^{s-1}e^{-t}\mathrm{d}t$ is the upper incomplete gamma function. The result in (10) can help the design of massive MIMO systems for the desired outage level. For example, we can derived how many users can be served simultaneously by the massive BS for a given γ_{th} value.

$$P_{out} \approx \frac{\sqrt{M}}{\sqrt{M}+K-2}\eta^{-(K-2)}\left[\int_{\frac{M+1}{\gamma_{th}}-\frac{1}{P_u}}^{\infty} e^{\frac{-\sqrt{M}}{\sqrt{M}+K-2}y}dy - \sum_{n=0}^{K-3}\eta^n\left(\frac{\sqrt{M}}{\sqrt{M}-1}\right)^n\int_{\frac{M+1}{\gamma_{th}}-\frac{1}{P_u}}^{\infty}\frac{y^n}{n!}e^{-\frac{\sqrt{M}}{\sqrt{M}-1}y}\mathrm{d}y\right] = \eta^{-(K-2)}e^{-\frac{\sqrt{M}}{\sqrt{M}+K-2}\left(\frac{M+1}{\gamma_{th}}-\frac{1}{P_u}\right)} - (1-\eta) \times\sum_{n=0}^{K-3}\frac{1}{n!}\eta^{n-K+2}\Gamma\left(n+1,\frac{\sqrt{M}}{\sqrt{M}-1}\left(\frac{M+1}{\gamma_{th}}-\frac{1}{P_u}\right)\right). \quad (10)$$

V. Simulation Results

In this section, we show simulation results to verify the derived results on the pdf of Y_k and the outage probability.

In Fig. 1, for a system with $M = 100$ and $K = 10, 30$, the simulated pdf (via Monte-Carlo simulation) of the interference power for User 1 Y_1 is shown and compared with the approximate pdf in (7) for $L = 10$, the approximation in (8), and the closed-form pdf in (9). This figure shows that (9) matches tightly with the simulation for all y range and both K values. The $L = 1$ approximation in (8) has significant offset to the left, thus underestimates the distribution tail. The $L = 10$ approximation has a better match than (8). But for $K = 30$, it also has

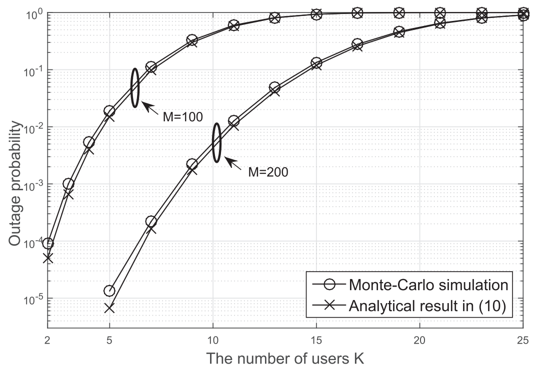

Fig. 2. Outage probability vs. K. $P_t = 10$ dB. $\gamma_{th} = 10$ dB.

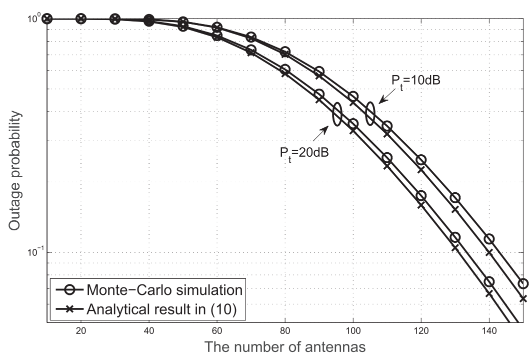

Fig. 3. Outage probability vs. M. $K = 10, \gamma_{th} = 10$ dB.

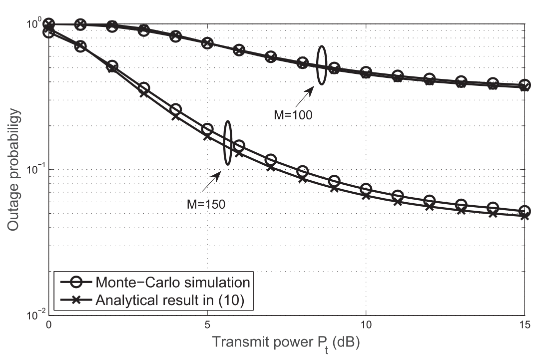

Fig. 4. Outage probability vs. P_t. $K = 10, \gamma_{th} = 10$ dB.

noticeable offset and underestimates the tail of the distribution as we have discussed in Section III.

Fig. 2 shows the outage probability for different number of users. Our analytical result in (10) has a tight match. When $M = 100$, $P_t = 10$ dB, $\gamma_{th} = 10$ dB, the outage probability is more than 10% when there are 7 users or more. Notice that on the other hand, the system sum-rate monotonically increases with K. This shows the importance of outage probability analysis. As K increases, even though the theoretical sum-rate increases, more users will be in outage and the actual system throughput can be very low.

Fig. 3 shows the outage probability for different number of antennas. It can be seen that the analytical result is accurate even for small M. Fig. 4 shows the outage probability for different transmit power. we can see that even when P_t increases, the outage probability does not decrease to zero. This is due to the user-interference, whose power increases with the transmit power. Both figures show that increasing the number of BS antennas can largely improve the outage probability performance.

VI. Conclusions

This paper analyzes the interference and outage probability of a single-cell multi-user massive MIMO system downlink with MF precoding. We derive the interference power distribution, and then obtain an analytical outage probability formula. Our analysis is different to the asymptotic deterministic equivalence analysis widely used in massive MIMO. The accuracy of the derived results is validated by simulations.

Appendix
Proof of Proposition 1

When $M \to \infty$, from the Lindeberg-Lévy central limit theorem, we have $\frac{1}{\sqrt{M}}\mathbf{h}_k\mathbf{h}_j^H \xrightarrow{d} \mathcal{CN}(0,1)$ for $k \neq j$, where $\xrightarrow{d}$ means convergence in distribution. Then $\frac{1}{M}|\mathbf{h}_k\mathbf{h}_j^H|^2$ converges to Gamma distribution $\phi(y;1,1)$. Next, we calculate the correlation coefficient of $\frac{1}{M}|\mathbf{h}_k\mathbf{h}_j^H|^2$ and $\frac{1}{M}|\mathbf{h}_k\mathbf{h}_l^H|^2$ for $j \neq l$, which is denoted as ρ_{jl}. Since $\mathbf{h}_k$'s are mutually independent, after tedious calculations, we can show that

$$\rho_{jl} = \frac{\mathrm{cov}\left(\frac{1}{M}|\mathbf{h}_k\mathbf{h}_j^H|^2, \frac{1}{M}|\mathbf{h}_k\mathbf{h}_l^H|^2\right)}{\sqrt{\mathrm{Var}\left\{\frac{1}{M}|\mathbf{h}_k\mathbf{h}_j^H|^2\right\}\mathrm{Var}\left\{\frac{1}{M}|\mathbf{h}_k\mathbf{h}_l^H|^2\right\}}} = \frac{1}{M}.$$

So $\frac{1}{M}\sum_{j=1,k\neq j}^{K}|\mathbf{h}_k\mathbf{h}_j^H|^2$ is a sum of $K-1$ correlated Gamma random variables with the same shape parameter of 1 and the same scale parameter of 1. The correlation coefficient is $1/M$. From Corollary 1 of [13], the pdf of $\frac{1}{M}\sum_{j=1,j\neq k}^{K}|\mathbf{h}_k\mathbf{h}_j^H|^2$ is

$$f_{Y_k}(y) = \prod_{i=1}^{K-1}\left(\frac{\sigma_1}{\sigma_i}\right)\sum_{j=0}^{\infty}\frac{\delta_j y^{K+j-2}e^{-y/\sigma_1}}{\sigma_1^{K+j-1}\Gamma(K+j-1)}, \tag{11}$$

where $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{K-1}$ are the ordered eigenvalues of the $(K-1)\times(K-1)$ matrix $\mathbf{A}$, whose diagonal entries are 1 and off-diagonal entries are $1/\sqrt{M}$, and δ_j's are defined iteratively as

$$\delta_0 \triangleq 1, \delta_{j+1} \triangleq \frac{1}{j+1}\sum_{m=1}^{j+1}\left[\sum_{n=1}^{K-1}\left(1-\frac{\sigma_1}{\sigma_n}\right)^m\right]\delta_{j+1-m}. \tag{12}$$

As $\mathbf{A}$ is a circulant matrix whose off-diagonal entries are the same, its eigenvalues can be calculated to be

$$\sigma_1 = \cdots = \sigma_{K-2} = 1 - \frac{1}{\sqrt{M}}, \sigma_{K-1} = 1 + \frac{K-2}{\sqrt{M}}. \tag{13}$$

Using (12)–(13), after some calculations, we have

$$\delta_j = \left(1 - \frac{\sqrt{M}-1}{\sqrt{M}+K-2}\right)^j. \tag{14}$$

Substituting (13) and (14) into (11), we conclude the proof.

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