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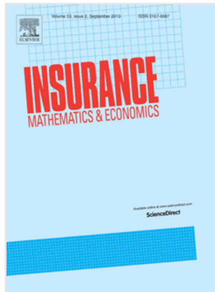

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The Finite-time Ruin Probability with Heavy-tailed and Dependent Insurance and Financial Risks

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Abstract

Consider a discrete-time insurance risk model in which the insurer makes both risk-free and risky investments. Assume that the one-period insurance and financial risks form a sequence of independent and identically distributed copies of a random pair (X, Y) with dependent components. When the product XY is heavy tailed, under a mild restriction on the dependence structure of (X, Y), we establish for the finite-time ruin probability an asymptotic formula, which coincides with the long-standing one in the literature. Various important special cases are presented, showing that our work generalizes and unifies some of recent ones.

Keywords: Asymptotics; Dependence; Finite-time ruin probability; Heavy-tailed distribution; Insurance and financial risks; Product

MSC 2010: Primary 91B30; Secondary: 62P05, 62E20, 62H20

1 Introduction

Consider a discrete-time insurance risk model. Within period i, $i \in \mathbb{N}$, the net insurance loss (equal to the total claim amount plus other costs minus the total premium income) is denoted by a real-valued random variable X_i. Suppose that the insurer makes both risk-free and risky investments, which lead to an overall stochastic discount factor over period i, denoted by a positive random variable Y_i. In the terminology of Norberg (1999) and Tang and Tsitsiashvili (2003), we call $\{X_i, i \in \mathbb{N}\}$ insurance risks and call $\{Y_i, i \in \mathbb{N}\}$ financial risks. Thus, the sum

$$\sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N},$$

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represents the stochastic present value of aggregate net losses up to time n. As is well known, the probability of ruin by time n, namely, the probability of the insurer's wealth process going below zero by time n, is equal to

$$\psi(x;n) = \Pr\left(\max_{1\le m\le n}\sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x\right), \qquad n \in \mathbb{N}, \tag{1.1}$$

where $x \ge 0$ is the initial wealth of the insurer.

Since Nyrhinen (1999, 2001) and Tang and Tsitsiashvili (2003, 2004), there has been a vast amount of works in the literature focusing on the asymptotic behavior of the ruin probability of this discrete-time risk model for heavy-tailed cases. This study is particularly relevant nowadays in view of recent more and more frequent natural, social and financial catastrophes causing the increasing prudence of regulators in the insurance and banking industries.

In Tang and Tsitsiashvili (2003, 2004), it was assumed that $\{X_i, i \in \mathbb{N}\}$ and $\{Y_i, i \in \mathbb{N}\}$ are two sequences of independent and identically distributed (i.i.d.) random variables and the two sequences are independent of each other as well. Undoubtedly, this assumption of complete independence is far unrealistic. Recently, a new trend of the study is to incorporate various dependence structures to the risk model. In this direction, we refer the reader to Goovaerts et al. (2005), Laeven et al. (2005), Tang and Vernic (2007), Chen and Ng (2007), Zhang et al. (2009), Weng et al. (2009), Chen (2011) and Yang and Wang (2013), among many others.

In particular, Chen (2011) extended the study to the situation where the insurance and financial risks are dependent according to a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution

$$\Lambda\left(x,y\right) = F(x)G(y)\left(1+\theta\overline{F}(x)\overline{G}(y)\right), \tag{1.2}$$

where $F = 1-\overline{F}$ on $\mathbb{R} = (-\infty,\infty)$ and $G = 1-\overline{G}$ on $\mathbb{R}_+ = [0,\infty)$ are marginal distribution functions, and $\theta \in [-1,1]$ is a parameter governing the strength of dependence. This is an important extension in view of the fact that the insurance risk X_i and the financial risk Y_i over period i occur in the same or a similar macroeconomic environment and, hence, should be strongly dependent on each other.

In this paper we shall follow Chen (2011) to assume that (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with a generic random pair (X, Y). We shall impose a mild restriction (3.1) on the dependence of (X, Y), which is much more general than the FGM structure. When the product XY is heavy tailed, or, more precisely, it has a long and dominatedly varying tail, we shall establish for the finite-time ruin probability an asymptotic formula

$$\psi(x;n) \sim \sum_{i=1}^{n} \Pr\left(X_i \prod_{j=1}^{i} Y_j > x\right). \tag{1.3}$$

This formula coincides with the long-standing one in the literature since Tang and Tsitsiashvili (2003). A few important special cases are presented as corollaries, showing that our work generalizes and unifies some of recent ones.

The rest of this paper consists of three sections. Section 2 prepares preliminaries of heavy-tailed distributions, Section 3 presents the main result and its corollaries, and Section 4 proves the main result.

2 Preliminaries

Throughout this paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \le 1$, write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$, and write $f(x) \asymp g(x)$ if $f(\cdot)$ and $g(\cdot)$ are weakly equivalent, that is, $0 < \liminf f(x)/g(x) \le \limsup f(x)/g(x) < \infty$. We often use the letter C to denote an absolute positive constant, which does not depend on the working variables such as x and A.

A distribution function H on $\mathbb{R}$ is said to be long tailed, written as $H \in \mathcal{L}$, if its ultimate right tail $\overline{H}$ satisfies

$$\lim_{x\to\infty} \frac{\overline{H}(x+y)}{\overline{H}(x)} = 1 \qquad \text{for all } y \in \mathbb{R}. \tag{2.1}$$

Automatically, relation (2.1) holds uniformly for every compact set of y. Hence, it is easy to see that there exists a function $a(\cdot)$, with $0 \le a(x) \le x/2$ and $a(x) \uparrow \infty$, such that relation (2.1) holds uniformly for $-a(x) \le y \le a(x)$; namely,

$$\lim_{x\to\infty} \sup_{-a(x)\le y\le a(x)} \left| \frac{\overline{H}(x+y)}{\overline{H}(x)} - 1 \right| = 0.$$

A distribution function H on $\mathbb{R}$ is said to be dominatedly-varying tailed, written as $H \in \mathcal{D}$, if its ultimate right tail $\overline{H}$ satisfies

$$\limsup_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} < \infty \qquad \text{for all } 0 < y < 1.$$

The intersection $\mathcal{L} \cap \mathcal{D}$ forms a useful subclass of heavy-tailed distributions and it has been often proposed as a standard assumption on the distributions of heavy-tailed risk variables. In particular, the intersection $\mathcal{L} \cap \mathcal{D}$ covers the famous class $\mathcal{R}$ of distributions with a regularly-varying tail. By definition, for a distribution function H on $\mathbb{R}$, we write $H \in \mathcal{R}_{-\alpha}$ for some $0 \le \alpha < \infty$ if its right tail $\overline{H}$ is regularly varying with index $-\alpha$; namely,

$$\lim_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} = y^{-\alpha} \qquad \text{for all } y > 0.$$

The reader is referred to the monograph Embrechts et al. (1997) for details of these and related classes of heavy-tailed distributions.

For a distribution function H with an ultimate right tail, define

$$M^*(H) = \inf\left\{-\frac{\log \overline{H}_*(y)}{\log y} : y > 1\right\} \quad \text{with} \quad \overline{H}_*(y) = \liminf_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)},$$

and we call $M^*(H)$ the upper Matuszewska index of H. It is clear that $H \in \mathcal{D}$ if and only if $0 \leq M^*(H) < \infty$, and that if $H \in \mathcal{R}_{-\alpha}$ then $M^*(H) = \alpha$.

For a distribution function H with $M^*(H) < \infty$, or, equivalently, $H \in \mathcal{D}$, by Proposition 2.2.1 of Bingham et at. (1987), we see that, for every $\beta > M^*(H)$, there are some positive constants C and D such that the inequality

$$\frac{\overline{H}(x)}{\overline{H}(xy)} \leq Cy^\beta \tag{2.2}$$

holds for all $xy \geq x \geq D$. From (2.2), it is easy to see that the relation

$$x^{-\beta} = o\left(\overline{H}(x)\right) \tag{2.3}$$

holds for every $\beta > M^*(H)$. See also Tang and Tsitsiashvili (2003) for these results.

Lemma 2.1 *Let X and Y be two independent random variables with Y positive. Denote by F and H the distribution functions of X and XY, respectively.*

(a) If $F \in \mathcal{L}\cap\mathcal{D}$ and $E[Y^\beta] < \infty$ for some $\beta > M^*(F)$, then $\overline{H}(x) \asymp \overline{F}(x)$, $H \in \mathcal{L}\cap\mathcal{D}$ and $M^*(H) = M^*(F)$.

(b) If $F \in \mathcal{R}_{-\alpha}$ and $E[Y^\beta] < \infty$ for some $\beta > \alpha \geq 0$, then $\overline{H}(x) \sim E[Y^\alpha]\overline{F}(x)$.

Proof. (a) By Theorem 2.2 of Cline and Samorodnitsky (1994), $H \in \mathcal{L}$. Moreover, by Theorem 3.3 of Cline and Samorodnitsky (1994), $H \in \mathcal{D}$ and $\overline{H}(x) \asymp \overline{F}(x)$. The result $M^*(H) = M^*(F)$ follows from $\overline{H}(x) \asymp \overline{F}(x)$.

(b) This is a restatement of Breiman's theorem; see Cline and Samorodnitsky (1994), who attributed the result to Breiman (1965). ■

3 The Main Result and Its Corollaries

Recall the insurance risk model introduced in Section 1. In the sequel, denote by F on $\mathbb{R}$, G on $\mathbb{R}_+$ and H on $\mathbb{R}$ the distribution functions of X, Y and XY, respectively. Here comes our main result:

Theorem 3.1 *Assume that $H \in \mathcal{L} \cap \mathcal{D}$, $E[Y^{\beta}] < \infty$ for some $\beta > M^*(H)$, and*

$$\lim_{A\to\infty} \limsup_{x\to\infty} \frac{\Pr(XY > x, Y > A)}{\Pr(XY > x)} = 0. \tag{3.1}$$

Then relation (1.3) holds for each $n \in \mathbb{N}$; namely,

$$\psi(x;n) \sim \sum_{i=1}^{n} \Pr\left(X_i \prod_{j=1}^{i} Y_j > x \right). \tag{1.3}$$

In Theorem 3.1, if $H \in \mathcal{R}_{-\alpha}$ and $E[Y^{\beta}] < \infty$ for some $\beta > \alpha \geq 0$, then applying Lemma 2.1(b), each term on the right-hand side of (1.3) satisfies

$$\Pr\left(X_i \prod_{j=1}^{i} Y_j > x \right) = \Pr\left(X_i Y_i \prod_{j=1}^{i-1} Y_j > x \right) \sim (E[Y^{\alpha}])^{i-1}\, \overline{H}(x).$$

Thus, relation (1.3) is simplified to

$$\psi(x;n) \sim \frac{1-(E[Y^{\alpha}])^{n}}{1-E[Y^{\alpha}]} \overline{H}(x), \tag{3.2}$$

where the right-hand side is understood as $n\overline{H}(x)$ if $E[Y^{\alpha}] = 1$. Furthermore, if $E[Y^{\alpha}] < 1$, then following the proof of Theorem 3.1 of Tang and Tsitsiashvili (2004), we can prove that relation (3.2) holds uniformly for all $n \in \mathbb{N}$; that is,

$$\lim_{x\to\infty} \sup_{n\in\mathbb{N}} \left| \frac{\psi(x;n)}{\frac{1-(E[Y^{\alpha}])^{n}}{1-E[Y^{\alpha}]} \overline{H}(x)} - 1 \right| = 0.$$

In particular, plugging $n = \infty$ into (3.2) yields

$$\psi(x;\infty) = \Pr\left(\max_{1\leq m<\infty} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x \right) \sim \frac{1}{1-E[Y^{\alpha}]} \overline{H}(x). \tag{3.3}$$

In this way, we obtain the following first corollary of Theorem 3.1:

Corollary 3.1 *If $H \in \mathcal{R}_{-\alpha}$, $E[Y^{\beta}] < \infty$ for some $\beta > \alpha \geq 0$, and condition (3.1) holds, then relation (3.2) holds for every $n \in \mathbb{N}$. Furthermore, if $E[Y^{\alpha}] < 1$, then relation (3.2) holds uniformly for all $n \in \mathbb{N}$ and, hence, relation (3.3) holds.*

We remark that the requirement in (3.1) is indeed mild and it provides us with enough flexibility in introducing dependence to (X, Y). We are going to present a few important special cases as corollaries of Theorem 3.1, with various concrete patterns of dependence ranging over independence, arbitrary dependence, FGM structure, common shock and regression-type dependence. In all of these corollaries, once the intersection $\mathcal{L} \cap \mathcal{D}$ shrinks to the class

$\mathcal{R}$, then, as has been done in Corollary 3.1, analogous simplifications and extensions of the obtained formulas follow immediately. In order to keep the paper short, we shall not repeat such simplifications and extensions.

First of all, note that if the financial risk Y is bounded, then the numerator in (3.1) is zero for all large A and relation (3.1) holds automatically regardless of any dependence structure between X and Y. This leads to the following:

Corollary 3.2 *If $F \in \mathcal{L} \cap \mathcal{D}$ and Y is bounded from above, then relation (1.3) holds for each $n \in \mathbb{N}$.*

Corollary 3.3 below retrieves Theorem 5.1 of Tang and Tsitsiashvili (2003):

Corollary 3.3 *If X and Y are independent, $F \in \mathcal{L} \cap \mathcal{D}$, and $E[Y^\beta] < \infty$ for some $\beta > M^*(F)$, then relation (1.3) holds for each $n \in \mathbb{N}$.*

Proof. By Lemma 2.1(a), we have $\overline{H}(x) \asymp \overline{F}(x)$, $H \in \mathcal{L} \cap \mathcal{D}$ and $M^*(H) = M^*(F)$. It remains to verify condition (3.1). Arbitrarily choosing $A > 1$ and $M^*(F)/\beta < p < 1$, by (2.2) and (2.3) we have

$$\begin{aligned} \Pr(XY > x, Y > A) &\leq \Pr(XY > x, x^p \geq Y > A) + \Pr(Y > x^p) \\ &\leq \int_A^{x^p} \overline{F}\left(\frac{x}{y}\right) dG(y) + \frac{1}{x^{p\beta}} E[Y^\beta] \\ &\lesssim C\overline{F}(x) \int_A^{x^p} y^\beta dG(y) + o\left(\overline{F}(x)\right) \\ &\lesssim C\overline{F}(x) E\left[Y^\beta 1_{(Y>A)}\right]. \end{aligned} \tag{3.4}$$

Then by the finiteness of $E[Y^\beta]$, condition (3.1) is verified. ■

Corollary 3.4 below partially retrieves Theorem 3.1 of Chen (2011) for the case of $\mathcal{L} \cap \mathcal{D}$ and its special case with $\theta = 0$ corresponds to Corollary 3.3:

Corollary 3.4 *If X and Y follow a bivariate FGM distribution function (1.2) with $F \in \mathcal{L} \cap \mathcal{D}$, $E[Y^\beta] < \infty$ for some $\beta > M^*(F)$, and $\theta \in (-1, 1]$, then relation (1.3) holds for each $n \in \mathbb{N}$.*

Proof. By relation (4.9) of Chen (2011, page 1041),

$$\Pr(XY > x) \sim \Pr(XZ_\theta > x), \tag{3.5}$$

where Z_θ is a nonnegative random variable independent of X and distributed by

$$G_\theta(y) = (1 - \theta) G(y) + \theta G^2(y), \qquad y > 0.$$

Thus, $E[Z_\theta^\beta] < \infty$. Then applying Lemma 2.1(a) to the right-hand side of (3.5), we have $\overline{H}(x) \asymp \overline{F}(x)$, $H \in \mathcal{L} \cap \mathcal{D}$ and $M^*(H) = M^*(F)$.

It remains to verify condition (3.1). Recall identity (4.5) of Chen (2011, page 1040),

$$\Lambda = (1+\theta)\, FG - \theta F G^2 - \theta F^2 G + \theta F^2 G^2.$$

Introduce six independent random variables X^*, X_1^*, X_2^*, Y^*, Y_1^* and Y_2^* with the first three identically distributed as X and the last three as Y. We have

$$\begin{aligned} & \Pr(XY > x, Y > A) \\ = \; & (1+\theta)\Pr(X^* Y^* > x, Y^* > A) - \theta \Pr(X^*(Y_1^* \vee Y_2^*) > x, Y_1^* \vee Y_2^* > A) \\ & -\theta \Pr((X_1^* \vee X_2^*)\, Y^* > x, Y^* > A) + \theta \Pr((X_1^* \vee X_2^*)(Y_1^* \vee Y_2^*) > x, Y_1^* \vee Y_2^* > A) \\ = \; & (1+\theta)\, I_1(x,A) - \theta I_2(x,A) - \theta I_3(x,A) + \theta I_4(x,A). \end{aligned} \tag{3.6}$$

Clearly, the conditions $F \in \mathcal{L}\cap\mathcal{D}$ and $E[Y^\beta] < \infty$ imply that $F^2 \in \mathcal{L}\cap\mathcal{D}$ and $E\left[(Y_1^* \vee Y_2^*)^\beta\right] < \infty$. By going along the same lines of (3.4), we have, respectively,

$$\begin{aligned} I_1(x,A) &\lesssim C\overline{F}(x) E\left[Y^\beta 1_{(Y>A)}\right], \\ I_2(x,A) &\lesssim C\overline{F}(x) E\left[(Y_1^* \vee Y_2^*)^\beta 1_{(Y_1^* \vee Y_2^* > A)}\right], \\ I_3(x,A) &\lesssim 2C\overline{F}(x) E\left[Y^\beta 1_{(Y>A)}\right], \\ I_4(x,A) &\lesssim 2C\overline{F}(x) E\left[(Y_1^* \vee Y_2^*)^\beta 1_{(Y_1^* \vee Y_2^* > A)}\right]. \end{aligned}$$

Substituting these estimates into (3.6) and recalling $\theta \in (-1, 1]$ and $\overline{H}(x) \asymp \overline{F}(x)$, it becomes straightforward to verify condition (3.1). ■

In Corollary 3.5 below we use another approach to model dependent risks. Let

$$X = UW \quad \text{and} \quad Y = VW, \tag{3.7}$$

where U, V and W are three independent random variables with U real valued while V and W positive. The incentive behind this modeling is that U and V are used to depict the magnitude of the risk variables, while W, often called common shock in contemporary credit risk modeling, is included to generate strong dependence for the risk variables. See for example Bassamboo et al. (2008) and Tang and Yuan (2013) for related discussions.

Corollary 3.5 *Let X and Y be modeled as in (3.7) with independent U, V and W. If U is distributed by $F_U \in \mathcal{L} \cap \mathcal{D}$ and $E[V^\beta] + E[W^{2\beta}] < \infty$ for some $\beta > M^*(F_U)$, then relation (1.3) holds for each $n \in \mathbb{N}$; that is,*

$$\psi(x;n) \sim \sum_{i=1}^n \Pr\left(U_i V_i W_i^2 \prod_{j=1}^{i-1} V_j W_j > x\right),$$

where $\{U, U_i, i \in \mathbb{N}\}$, $\{V, V_i, i \in \mathbb{N}\}$ and $\{W, W_i, i \in \mathbb{N}\}$ are three independent sequences of i.i.d. random variables.

Proof. Note that $\overline{H}(x) = \Pr(UVW^2 > x)$, applying Lemma 2.1(a), we have $\overline{H}(x) \asymp \overline{F_U}(x)$, $H \in \mathcal{L} \cap \mathcal{D}$ and $M^*(H) = M^*(F_U)$. It remains to verify condition (3.1). Analogously to (3.4), there is some $C > 0$ such that

$$\Pr(XY > x, Y > A) = \Pr\left(UVW^2 > x, VW > A\right) \lesssim C\overline{F_U}(x)E[V^\beta W^{2\beta}]1_{(VW>A)}.$$

Condition (3.1) is verified and, hence, relation (1.3) holds for each $n \in \mathbb{N}$. ■

Motivated by the works of Asimit and Badescu (2010), Li et al. (2010) and Chen and Yuen (2012), we make the following regression-type assumption on the dependence between X and Y: there is a measurable function $h : [0, \infty) \longmapsto (0, \infty)$ such that the relation

$$\Pr(X > x | Y = y) \sim \Pr(X > x)\, h(y) \tag{3.8}$$

holds uniformly for $y \geq 0$; namely,

$$\lim_{x\to\infty} \sup_{y\geq 0} \left| \frac{\Pr(X > x|Y = y)}{\Pr(X > x)\, h(y)} - 1 \right| = 0.$$

Corollary 3.6 *Under condition (3.8), if $F \in \mathcal{L} \cap \mathcal{D}$ and $E\left[Y^\beta (h(Y) + 1)\right] < \infty$ for some $\beta > M^*(F)$, then relation (1.3) holds for each $n \in \mathbb{N}$.*

Proof. A natural consequence of condition (3.8) is that $E[h(Y)] = 1$; see also Li et al. (2010). Introduce a nonnegative random variable $\tilde{Y}$ independent of X and distributed by

$$\Pr\left(\tilde{Y} \in dy\right) = h(y) \Pr(Y \in dy).$$

Under condition (3.8), with arbitrarily chosen $M^*(F)/\beta < p < 1$ we have

$$\begin{aligned} \overline{H}(x) &= \Pr(XY > x, Y \leq x^p) + \Pr(XY > x, Y > x^p) \\ &= \int_0^{x^p} \Pr\left(X > \frac{x}{y} \middle| Y = y\right) \Pr(Y \in dy) + O(\Pr(Y > x^p)) \\ &\sim \int_0^{x^p} \Pr\left(X > \frac{x}{y}\right) h(y) \Pr(Y \in dy) + O\left(x^{-p\beta}\right) \\ &= \left(\int_0^\infty - \int_{x^p}^\infty\right) \Pr\left(X > \frac{x}{y}\right) \Pr\left(\tilde{Y} \in dy\right) + O\left(x^{-p\beta}\right) \\ &= \Pr\left(X\tilde{Y} > x\right) + O\left(x^{-p\beta}\right), \end{aligned} \tag{3.9}$$

where in the second and third steps we applied $E[Y^\beta] < \infty$ and

$$E\left[\tilde{Y}^\beta\right] = E\left[Y^\beta h(Y)\right] < \infty.$$

Applying Lemma 2.1(a) to the right-hand side of (3.9), and recalling (2.3), we have

$$\overline{H}(x) \sim \Pr\left(X\tilde{Y} > x\right) + O\left(x^{-p\beta}\right) \sim \Pr\left(X\tilde{Y} > x\right) \asymp \overline{F}(x).$$

Consequently, $H \in \mathcal{L} \cap \mathcal{D}$ and $M^*(H) = M^*(F)$.

It remains to verify condition (3.1). Analogously to (3.9) and (3.4), with arbitrarily chosen $M^*(F)/\beta < p < 1$, some $C > 0$ and all large x, we have

$$\begin{aligned}
& \Pr\left(XY > x, Y > A\right) \\
\sim\; & \int_A^{x^p} \Pr\left(X > \frac{x}{y}\right) h(y) \Pr\left(Y \in dy\right) + O\left(x^{-p\beta}\right) \\
=\; & \int_A^{x^p} \Pr\left(X > \frac{x}{y}\right) \Pr\left(\tilde{Y} \in dy\right) + o\left(\overline{F}(x)\right) \\
\leq\; & C\overline{F}(x) \int_A^{x^p} y^\beta \Pr\left(\tilde{Y} \in dy\right) + o\left(\overline{F}(x)\right) \\
\leq\; & C\overline{F}(x) E\left[\tilde{Y}^\beta 1_{(\tilde{Y} > A)}\right] + o\left(\overline{F}(x)\right).
\end{aligned}$$

Condition (3.1) is verified and, hence, relation (1.3) holds for each $n \in \mathbb{N}$. ■

4 Proof of the Main Result

To help with presentation, we introduce the notation $\Pi_0 = 1$ and $\Pi_i = \prod_{j=1}^{i} Y_j$ for $i \in \mathbb{N}$.

4.1 A lemma

The following lemma plays a key role in the proof of Theorem 3.1:

Lemma 4.1 *Under the conditions of Theorem 3.1, it holds for every positive function $a(x) \to \infty$ and for each $1 \leq i \neq k \leq n$ that*

$$\Pr\left(|X_i| \Pi_i > a(x), X_k \Pi_k > x\right) = o\left(\overline{H}(x)\right). \tag{4.1}$$

Proof. First consider $1 \leq i < k \leq n$. With some $M^*(H)/\beta < p < 1$, we do the split

$$\begin{aligned}
& \Pr\left(|X_i| \Pi_i > a(x), X_k \Pi_k > x\right) \\
=\; & \Pr\left(|X_i| \Pi_i > a(x), X_k \Pi_k > x, \Pi_{k-1} > x^p\right) \\
& + \Pr\left(|X_i| \Pi_i > a(x), X_k \Pi_k > x, \Pi_{k-1} \leq x^p\right) \\
=\; & I_1(x) + I_2(x).
\end{aligned}$$

For $I_1(x)$, by Markov's inequality and relation (2.3),

$$I_1(x) \leq \Pr\left(\Pi_{k-1} > x^p\right) \leq \frac{1}{x^{p\beta}} \left(E[Y^\beta]\right)^{k-1} = o\left(\overline{H}(x)\right).$$

For $I_2(x)$, noticing that $\{X_i, Y_1, \ldots, Y_{k-1}\}$ are independent of $\{X_k, Y_k\}$ and applying inequality (2.2), it holds for some $C > 1$ and all large x that

$$\begin{aligned} I_2(x) &= \int_0^{x^p} \Pr\left(X_k Y_k > \frac{x}{y}\right) \Pr\left(|X_i| \Pi_i > a(x), \Pi_{k-1} \in dy\right) \\ &\leq \int_0^{x^p} \Pr\left(X_k Y_k > \frac{x}{y \vee 1}\right) \Pr\left(|X_i| \Pi_i > a(x), \Pi_{k-1} \in dy\right) \\ &\leq C\overline{H}(x) \int_0^{x^p} (y \vee 1)^\beta \Pr\left(|X_i| \Pi_i > a(x), \Pi_{k-1} \in dy\right) \\ &\leq C\overline{H}(x) E\left[\left(\Pi_{k-1}^\beta \vee 1\right) 1_{(|X_i|\Pi_i > a(x))}\right] \\ &= o\left(\overline{H}(x)\right), \end{aligned} \tag{4.2}$$

where the last step is due to

$$E\left[\Pi_{k-1}^\beta \vee 1\right] \leq E\left[\prod_{j=1}^{k-1}\left(Y_j^\beta \vee 1\right)\right] = \left(E\left[Y^\beta \vee 1\right]\right)^{k-1} < \infty$$

and the obvious fact that $\Pr\left(|X_i| \Pi_i > a(x)\right) \to 0$. Thus, relation (4.1) holds.

Next consider $1 \leq k < i \leq n$. Arbitrarily choosing some $A > 0$ and using the same p as specified above, we derive

$$\begin{aligned} \Pr\left(|X_i| \Pi_i > a(x), X_k\Pi_k > x\right) &= \Pr\left(|X_i| \Pi_i > a(x), X_k\Pi_k > x, \Pi_{k-1} > x^p\right) \\ &\quad + \Pr\left(|X_i| \Pi_i > a(x), X_k\Pi_k > x, Y_k > A, \Pi_{k-1} \leq x^p\right) \\ &\quad + \Pr\left(|X_i| \Pi_i > a(x), X_k\Pi_k > x, Y_k \leq A, \Pi_{k-1} \leq x^p\right) \\ &= J_1(x) + J_2(x) + J_3(x). \end{aligned} \tag{4.3}$$

Similarly to $I_1(x)$, it holds that

$$J_1(x) \leq \Pr\left(\Pi_{k-1} > x^p\right) = o\left(\overline{H}(x)\right).$$

Now we turn to $J_2(x)$. By condition (3.1), for arbitrarily given $\varepsilon > 0$, it holds for all large A and x that

$$\Pr\left(XY > x, Y > A\right) \leq \varepsilon\overline{H}(x).$$

By this and inequality (2.2), it holds for some $C > 1$, all large A and x that

$$\begin{aligned} J_2(x) &\leq \Pr\left(X_k\Pi_k > x, Y_k > A, \Pi_{k-1} \leq x^p\right) \\ &= \int_0^{x^p} \Pr\left(X_k Y_k > \frac{x}{y}, Y_k > A\right) \Pr\left(\Pi_{k-1} \in dy\right) \\ &\leq \varepsilon \int_0^{x^p} \Pr\left(X_k Y_k > \frac{x}{y}\right) \Pr\left(\Pi_{k-1} \in dy\right) \\ &\leq C\varepsilon\overline{H}(x) \int_0^{x^p} (y \vee 1)^\beta \Pr\left(\Pi_{k-1} \in dy\right) \\ &\leq C\varepsilon\overline{H}(x) E\left[\Pi_{k-1}^\beta \vee 1\right]. \end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ and the finiteness of the last expectation, it follows that

$$\lim_{A\to\infty} \limsup_{x\to\infty} \frac{J_2(x)}{\overline{H}(x)} = 0.$$

Finally, we deal with $J_3(x)$. Clearly,

$$J_3(x) \le \Pr\left(|X_i| \prod_{j=1, j\neq k}^{i} Y_j > \frac{a(x)}{A}, X_k \Pi_k > x, \Pi_{k-1} \le x^p \right).$$

Then, analogously to (4.2), it holds for some $C > 1$ that

$$\begin{aligned} J_3(x) &\le \int_0^{x^p} \Pr\left(X_k Y_k > \frac{x}{y} \right) \Pr\left(|X_i| \prod_{j=1, j\neq k}^{i} Y_j > \frac{a(x)}{A}, \Pi_{k-1} \in dy \right) \\ &\le C\overline{H}(x) E\left[\left(\Pi_{k-1}^{\beta} \vee 1 \right) 1_{\left(|X_i| \prod_{j=1, j\neq k}^{i} Y_j > a(x)/A \right)} \right] \\ &= o\left(\overline{H}(x) \right). \end{aligned}$$

Plugging these estimates for $J_1(x)$, $J_2(x)$ and $J_3(x)$ into (4.3) we conclude that relation (4.1) still holds. ■

A referee kindly points out to us, and we agree, that Lemma 4.1 can also be obtained by using Lemma 7 of a recently published paper by Tang and Yuan (2014).

4.2 Proof of Theorem 3.1

Note that, by (1.1),

$$\Pr\left(\sum_{i=1}^{n} X_i \Pi_i > x \right) \le \psi(x; n) \le \Pr\left(\sum_{i=1}^{n} X_i^{+} \Pi_i > x \right),$$

where each $X_i^+ = X_i \vee 0$ denotes the positive part of X_i. In order to prove (1.3), it suffices to prove the following two relations:

$$\Pr\left(\sum_{i=1}^{n} X_i^{+} \Pi_i > x \right) \lesssim \sum_{i=1}^{n} \Pr\left(X_i \Pi_i > x \right); \tag{4.4}$$

$$\Pr\left(\sum_{i=1}^{n} X_i \Pi_i > x \right) \gtrsim \sum_{i=1}^{n} \Pr\left(X_i \Pi_i > x \right). \tag{4.5}$$

By Lemma 2.1(a), each $X_i \Pi_i = X_i Y_i \Pi_{i-1}$ follows a distribution in $\mathcal{L} \cap \mathcal{D}$. Thus, we can find a function $a(\cdot) : [0, \infty) \to [0, \infty)$ with $a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$ such that, for all $i = 1, \dots, n$,

$$\Pr\left(X_i \Pi_i > x \pm a(x) \right) \sim \Pr\left(X_i \Pi_i > x \right). \tag{4.6}$$

We have

$$\begin{aligned}
\Pr\left(\sum_{i=1}^{n} X_i^+\Pi_i > x\right) &= \Pr\left(\sum_{i=1}^{n} X_i^+\Pi_i > x, \bigcup_{i=1}^{n}\left(X_i^+\Pi_i > x - a(x)\right)\right) \\
&\quad + \Pr\left(\sum_{i=1}^{n} X_i^+\Pi_i > x, \bigcap_{i=1}^{n}\left(X_i^+\Pi_i \le x - a(x)\right)\right) \\
&= I_1(x) + I_2(x). \qquad (4.7)
\end{aligned}$$

By (4.6),

$$I_1(x) \le \sum_{i=1}^{n} \Pr\left(X_i\Pi_i > x - a(x)\right) \sim \sum_{i=1}^{n} \Pr\left(X_i\Pi_i > x\right).$$

For $I_2(x)$, by Lemma 4.1 we have

$$\begin{aligned}
I_2(x) &= \sum_{k=1}^{n} \Pr\left(\sum_{i=1}^{n} X_i^+\Pi_i > x, X_k^+\Pi_k > \frac{x}{n}, \bigcap_{i=1}^{n}\left(X_i^+\Pi_i \le x - a(x)\right)\right) \\
&\le \sum_{k=1}^{n} \Pr\left(\sum_{i=1, i\neq k}^{n} X_i^+\Pi_i > a(x), \frac{x}{n} < X_k^+\Pi_k \le x - a(x)\right) \\
&\le \sum_{k=1}^{n}\sum_{i=1, i\neq k}^{n} \Pr\left(X_i\Pi_i > \frac{a(x)}{n-1}, X_k\Pi_k > \frac{x}{n}\right) \\
&= o\left(\Pr\left(XY > \frac{x}{n}\right)\right) \\
&= o\left(\overline{H}(x)\right).
\end{aligned}$$

Plugging these estimates for $I_1(x)$ and $I_2(x)$ into (4.7) yields relation (4.4).

Now we prove relation (4.5). By Bonferroni's inequality we have

$$\begin{aligned}
\Pr\left(\sum_{i=1}^{n} X_i\Pi_i > x\right) &\ge \Pr\left(\sum_{i=1}^{n} X_i\Pi_i > x, \bigcup_{k=1}^{n}\left(X_k\Pi_k > x + a(x)\right)\right) \\
&\ge \sum_{k=1}^{n} \Pr\left(\sum_{i=1}^{n} X_i\Pi_i > x, X_k\Pi_k > x + a(x)\right) \\
&\quad - \sum_{1\le k\neq l\le n} \Pr\left(\sum_{i=1}^{n} X_i\Pi_i > x, X_k\Pi_k > x + a(x), X_l\Pi_l > x + a(x)\right) \\
&\ge \sum_{k=1}^{n} \Pr\left(X_k\Pi_k > x + a(x)\right) \\
&\quad - \sum_{k=1}^{n} \Pr\left(\sum_{i=1}^{n} X_i\Pi_i \le x, X_k\Pi_k > x + a(x)\right) \\
&\quad - \sum_{1\le k\neq l\le n} \Pr\left(X_k\Pi_k > x + a(x), X_l\Pi_l > x + a(x)\right) \\
&= J_1(x) - J_2(x) - J_3(x). \qquad (4.8)
\end{aligned}$$

By (4.6),

$$J_1(x) \sim \sum_{k=1}^{n} \Pr\left(X_k \Pi_k > x\right).$$

For each term in $J_2(x)$, by Lemma 4.1 we have

$$\begin{aligned}
& \Pr\left(\sum_{i=1}^{n} X_i \Pi_i \le x, X_k \Pi_k > x + a(x)\right) \\
\le\ & \Pr\left(\sum_{i=1, i\neq k}^{n} X_i \Pi_i \le -a(x), X_k \Pi_k > x + a(x)\right) \\
\le\ & \sum_{i=1, i\neq k}^{n} \Pr\left(X_i \Pi_i \le -\frac{a(x)}{n-1}, X_k \Pi_k > x + a(x)\right) \\
=\ & o\left(\overline{H}(x)\right).
\end{aligned}$$

Thus,

$$J_2(x) = o\left(\overline{H}(x)\right).$$

It follows straightforwardly from Lemma 4.1 too that

$$J_3(x) = o\left(\overline{H}(x)\right).$$

Plugging these estimates for $J_1(x)$, $J_2(x)$ and $J_3(x)$ into (4.8) yields relation (4.5).

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