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Possibility linear programming with trapezoidal fuzzy numbers

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ABSTRACT

Fuzzy linear programming with trapezoidal fuzzy numbers (TrFNs) is considered and a new method is developed to solve it. In this method, TrFNs are used to capture imprecise or uncertain information for the imprecise objective coefficients and/or the imprecise technological coefficients and/or available resources. The auxiliary multi-objective programming is constructed to solve the corresponding possibility linear programming with TrFNs. The auxiliary multi-objective programming involves four objectives: minimizing the left spread, maximizing the right spread, maximizing the left endpoint of the mode and maximizing the middle point of the mode. Three approaches are proposed to solve the constructed auxiliary multi-objective programming, including optimistic approach, pessimistic approach and linear sum approach based on membership function. An investment example and a transportation problem are presented to demonstrate the implementation process of this method. The comparison analysis shows that the fuzzy linear programming with TrFNs developed in this paper generalizes the possibility linear programming with triangular fuzzy numbers.

1. Introduction

In the transportation problems, sometimes the unit cost of transportation may be expressed by fuzzy number [1] other than the real number because of the influence by the various subjective and objective factors. Consequently, the optimization model minimizing the total cost is constructed as a fuzzy mathematical programming. In the investment problem, the interest incomes of money market accounts are changed with market and may not be precise as time progresses. The optimization model maximizing the total net worth is also formulated as a fuzzy mathematical programming [2]. Therefore, the fuzzy mathematical programming is of a great importance for scientific researches and real applications.

The research on fuzzy mathematical programming has been an active area since Bellman and Zadeh [1] proposed the definition of fuzzy decision making [2–21]. The fuzzy mathematical programming was widely applied to many fields, such as transportation plan [12,13], supply chain network [15,16], dynamic virtual hub location [17], investment problem [2]. All these fuzzy mathematical (or linear) programming models can be classified into the following four categories according to the types of the fuzzy numbers.

The first is fuzzy programming model with intervals. For example, Ishibuchi and Tanaka [20] proposed the multi-objective programming in optimization of the interval objective function. They gave the definitions of the maximization and minimization problems with the interval objective functions. The second is fuzzy linear programming model with

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triangular fuzzy numbers (TFNs) [2,18,19]. For example, Lai and Hwang [2] developed a new approach to some possibilistic linear programming problems with TFNs. They transformed the fuzzy linear programming into a multi-objective linear programming model, involving three objective functions: minimizing the low loss, maximizing the most possible value and maximizing the upper the profit. The third is fuzzy linear programming model with trapezoidal fuzzy numbers (TrFNs) [5–11]. For example, Ganesan and Veeramani [8] and Ebrahimnejad [11] studied the fuzzy linear programs with TrFNs, but the constructed fuzzy linear programming models are only suitable for the symmetrical TrFNs. Maleki et al. [5–7], Allahviranloo et al. [9] and Liu [10] utilized the ranking function to solve the fuzzy linear programming models with TrFNs. The last is the optimization in an intuitionistic fuzzy environment [12,13]. For example, Angelove [12] applied the degrees of rejection of constraints and values of the objective to formulate the optimization problem in an intuitionistic fuzzy environment and transformed into the crisp optimization problem. Dubey et al. [13] proposed optimistic approach, pessimistic approach and mixed approach to solve the intuitionistic fuzzy linear programming problem.

Though fuzzy linear programming models with TrFNs have been researched in [5–11], the fuzzy linear programming models studied in [8,11] are only suitable for the symmetrical TrFNs, which significantly restricts the application scope of the models. In [5–7,9,10], the different ranking methods of TrFNs may result in the different crisp linear programming models, thus the obtained optimal solutions may be changed with the chosen ranking methods of TrFNs. Therefore, it is necessary to develop a reliable and stable method to solve the fuzzy linear programming models with TrFNs.

Since TrFN permits two parameters to represent the most possible values, while TFN uses the single parameter to represent the most possible value, TFN is a special case of TrFN. Therefore, TrFN is valuable both for modeling imprecision and for its ability to easily reflect the ambiguous nature of subjective judgments. The aim of this paper is to extend the possibility linear programming with TFNs [2] to develop a new possibility linear programming with TrFNs. In this method, TrFNs are used to capture imprecise or uncertain information for the imprecise objective coefficients and/or the imprecise technological coefficients and/or resources. The auxiliary multi-objective programming is constructed to solve the corresponding possibility linear programming with TrFNs. The auxiliary multi-objective programming involves four objectives: minimizing the left spread, maximizing the right spread, maximizing the left endpoint of the mode and maximizing the middle point of the mode.

We note that a similar idea can be found in the possibility linear programming method introduced by Lai and Hwang [2]. However, significant differences in features exist between the two developed methodologies. First, Lai and Hwang [2] studied the possibility linear programming with TFNs, i.e., the objective coefficients, technological coefficients, and available resources are TFNs other than TrFNs. In contrast, this paper utilizes TrFNs to represent these imprecise variables and proposed the method for solving the possibility linear programming with TrFNs. Since TFNs are a special case of TrFNs, the possibility linear programming with TFNs proposed by Lai and Hwang [2] is just a special case of the one with TrFNs proposed in this paper. Second, Lai and Hwang [2] only adapted the Zimmermam's fuzzy programming method [21] to solve the multi-objective linear programming, while this paper proposes three kinds of approaches to solving the constructed auxiliary multi-objective programming, i.e., the pessimistic approach, optimistic approach, and linear sum approach based on membership function. These approaches greatly enhance the flexibility of the proposed method for different decision makers (DMs). Third, to construct the membership functions of the objective functions, Lai and Hwang [2] need solve six linear programming models to obtain the positive and negative ideal solutions. Conversely, in this paper the maximum objective values can be obtained by solving four linear programming models and thereby the minimum objective values are determined by directly comparing the objective values. Thus, the proposed method in this paper need less computation cost than the method in [2].

The paper is organized as follows. In Section 2, the definition of TrFN is defined and the interval objective programming is introduced. In Section 3, the possibility linear programming with TrFNs is developed. The proposed possibility linear programming method is illustrated with an investment problem and comparison analysis is conducted in Section 4. A potential application to transportation problem is give in Section 5. Concluding remark is given in Section 6.

2. Definition for trapezoidal fuzzy numbers and interval objective programming

2.1. Definition for trapezoidal fuzzy numbers

A fuzzy number $\tilde{m}$ is a special fuzzy subset on the set R of real numbers. Let $\tilde{m} = (l, m_1, m_2, r)$ be a TrFN, where the membership function $\mu_{\tilde{m}}$ of $\tilde{m}$ is

$$\mu_{\tilde{m}}(x) = \begin{cases} \frac{x-l}{m_1-l} & (l \leqslant x < m_1), \\ 1 & (m_1 \leqslant x \leqslant m_2), \\ \frac{r-x}{r-m_2} & (m_2 < x \leqslant r). \end{cases}$$

The closed interval $[m_1, m_2]$ is the mode of $\tilde{m}$. l and r are the lower and upper limits of $\tilde{m}$ [22].

It is easily seen that a TrFN $\tilde{m} = (l, m_1, m_2, r)$ is reduced to a real number m if $l = m_1 = m_2 = r$. Conversely, a real number m can be written as a TrFN $\tilde{m} = (m, m, m, m)$. A TrFN $\tilde{m} = (l, m_1, m_2, r)$ is reduced to a TrFN $\tilde{m} = (l, m_1, r)$ if $m_1 = m_2$.

TrFN $\tilde{m} = (l, m_1, m_2, r)$ is called a positive TrFN if $l \geqslant 0$ and one of l, m_1, m_2 and r is non-zero. Furthermore, $\tilde{m} = (l, m_1, m_2, r)$ is called a normalized positive TrFN if it is a positive TrFN, $l \geqslant 0$ and $r \leqslant 1$.

2.2. Interval objective programming

Ishibuchi and Tanaka [20] gave the definitions of the maximization and minimization problems with the interval objective functions, which are introduced in Definitions 1 and 2 as follows.

Definition 1 [20]. Let $\tilde{a} = [a_l, a_u]$ be an interval. The maximization problem with the interval objective function is described as follows:

$$\max \{\tilde{a}\},$$
$$s.t.\ \tilde{a} \in \Omega,$$

which is equivalent to the following bi-objective mathematical programming problem:

$$\max \left\{a_l, \frac{1}{2}(a_l + a_u)\right\},$$
$$s.t.\ \tilde{a} \in \Omega,$$

where Ω is a set of constraints in which the variable $\tilde{a}$ should satisfy according to requirements in real situations.

Definition 2 [20]. Let $\tilde{a} = [a_l, a_u]$ be an interval. The minimization problem with the interval objective function is described as follows:

$$\min \{\tilde{a}\},$$
$$s.t.\ \tilde{a} \in \Omega,$$

which is equivalent to the following bi-objective mathematical programming problem:

$$\min \left\{a_u, \frac{1}{2}(a_l + a_u)\right\},$$
$$s.t.\ \tilde{a} \in \Omega.$$

3. Possibility linear programming with trapezoidal fuzzy numbers

Lai and Hwang [2] studied the possibility linear programming with TFNs. In the sequent, we generalize and extend the possibility linear programming to suit the case of TrFNs.

Without loss of generality, we define a possibility linear programming with TrFNs as follows:

$$\max \{\tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}\},$$
$$s.t. \begin{cases} \tilde{\mathbf{A}}x \leqslant \tilde{\mathbf{b}}, \\ \mathbf{x} \geqslant \mathbf{0}. \end{cases} \tag{1}$$

where $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$ is the technological coefficient matrix, $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)^{\mathrm{T}}$ is the available resource vector, and $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)^{\mathrm{T}}$ is the objective coefficient vector, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the decision variable vector. $\tilde{a}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}), \tilde{c}_j = (c_{jl}, c_{jm_1}, c_{jm_1}, c_{jr})$ and $\tilde{b}_i = (b_{il}, b_{im_1}, b_{im_2}, b_{ir})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are all known TrFNs. $x_j \geqslant 0 (j = 1, 2, \ldots, n)$ is crisp and unknown, which is to be solved.

For simplicity, some notations are introduced as follows:

$$\boldsymbol{A}_l = (a_{ij_l})_{m\times n}, \quad \boldsymbol{A}_{m_1} = (a_{ij_{m_1}})_{m\times n}, \quad \boldsymbol{A}_{m_2} = (a_{ij_{m_2}})_{m\times n}, \quad \boldsymbol{A}_r = (a_{ijr})_{m\times n},$$

$$\boldsymbol{c}_l = (c_{1l}, c_{2l}, \ldots, c_{nl})^T, \quad \boldsymbol{c}_{m_1} = (c_{1m_1}, c_{2m_1}, \ldots, c_{nm_1})^T, \quad \boldsymbol{c}_{m_2} = (c_{1m_2}, c_{2m_2}, \ldots, c_{nm_2})^T, \quad \boldsymbol{c}_r = (c_{1r}, c_{2r}, \ldots, c_{nr})^T,$$

$$b_l = (b_{1l}, b_{2l}, \ldots, b_{ml})^T, \quad b_{m_1} = (b_{1m_1}, b_{2m_1}, \ldots, b_{mm_1})^T, \quad b_{m_2} = (b_{1m_2}, b_{2m_2}, \ldots, b_{mm_2})^T, \quad b_r = (b_{1r}, b_{2r}, \ldots, b_{mr})^T.$$

3.1. Imprecise objective coefficient $\tilde{c}$

First, suppose that in Eq. (1) $\tilde{\mathbf{c}}$ is imprecise objective coefficient, whereas $\tilde{\mathbf{A}}$ is crisp technological coefficient matrix and $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)^{\mathrm{T}}$ is crisp available resource vector. That is, Eq. (1) can be rewritten as follows:

$$\max \{\tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}\},$$
$$s.t. \begin{cases} \mathbf{Ax} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases} \tag{2}$$

where $\mathbf{A} = (a_{ij})_{m\times n}, \mathbf{b} = (b_1, b_2, \ldots, b_m)^{\mathrm{T}}, a_{ij} \in R, b_i \in R (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.

In Eq. (2), the fuzzy objective $\tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}$ is a TrFN $\tilde{z} = \left((\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}\right)$. The most possible value of $\tilde{z}$ is the interval $[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}]$, the lower and upper are $(\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}$ and $(\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}$, respectively. Intuitively, maximization the fuzzy objective $\tilde{z}$ can be obtained by maximizing the lower $(\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}$, upper $(\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}$, and the two endpoints $(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}$ and $(\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}$ of the mode $[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}]$ simultaneously. Therefore, Eq. (2) can be solved by the following multi-objective programming:

$$\begin{aligned} &\max\ \{(\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}\}, \\ &s.t. \begin{cases} \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}. \end{cases} \end{aligned} \tag{3}$$

However, the above four objective functions $(\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}$ and $(\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}$ should always preserve the form of the TrFN $\left((\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}\right)$ during the optimization process. Thus, in order to keep the TrFN shape (normal and convex) of the possibility distribution, we change the above four objective functions in an effective way.

For the mode of TrFN $\left((\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}\right)$, since the objective function of Eq. (2) is to maximize $\{\tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}\}$, it is natural to maximize the interval $[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}]$ for this objective function. According to Definition 1, in order to maximize the interval $[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}]$, we need maximize the left endpoint $(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}$ and maximize the middle point $\frac{1}{2}[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} + (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}]$ of this interval simultaneously. For the lower and upper limits of TrFN $\left((\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}, (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}\right)$, we minimize $[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}]$ and maximize $[(\mathbf{c}_r)^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}]$ instead of maximizing the lower $(\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}$ and the upper $(\mathbf{c}_r)^{\mathrm{T}}\mathbf{x}$, respectively. This analysis can be illustrated with the following Fig. 1.

It can be seen from Fig. 1 that Eq. (3) can be transformed into the following multi-objective programming model:

$$\begin{aligned} &\min\ \{z_1 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}\}, \\ &\max\ \{z_2 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}\}, \\ &\max\ \{z_3 = \frac{1}{2}[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} + (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x})]\}, \\ &\max\ \{z_4 = (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}\}, \\ &s.t. \begin{cases} \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}. \end{cases} \end{aligned} \tag{4}$$

Although Eq. (4) is a also multi-objective linear programming model, it can effectively keep the TrFN shape of objective function $\tilde{z}$. There are few standard ways of defining a solution of multi-objective programming. Normally, the concept of Pareto optimal/efficient solutions is commonly-used. Next, we propose three kinds of approaches to solving this multi-objective linear programming model.

Since the objective function z_i is the function of the decision variable vector $\mathbf{x}$, simply denote by $z_i = z_i(\mathbf{x})(i = 1, 2, 3, 4)$. Let $z_i^{\max}$ and $\mathbf{x}_i^*$ respectively be the maximum objective value and the optimal solution for the following single objective linear programming model:

$$\begin{aligned} &\max\ \{z_i = z_i(x)\}, \\ &s.t. \begin{cases} \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}. \end{cases} \end{aligned} \tag{5}$$

Then, set $z_i^{\min} = \min\{z_i(\mathbf{x}_1^*), z_i(\mathbf{x}_2^*), z_i(\mathbf{x}_3^*), z_i(\mathbf{x}_4^*)\}(i = 1, 2, 3, 4)$. The linear membership function of the objective function z_1 can be calculated as follows:

$$\mu_{z_1}(\mathbf{x}) = \begin{cases} 1 & \text{if } z_1 < z_1^{\min}, \\ (z_1^{\max} - z_1)/(z_1^{\max} - z_1^{\min}) & \text{if } z_1^{\min} \leqslant z_1 \leqslant z_1^{\max}, \\ 0 & \text{if } z_1 > z_1^{\max}, \end{cases} \tag{6}$$

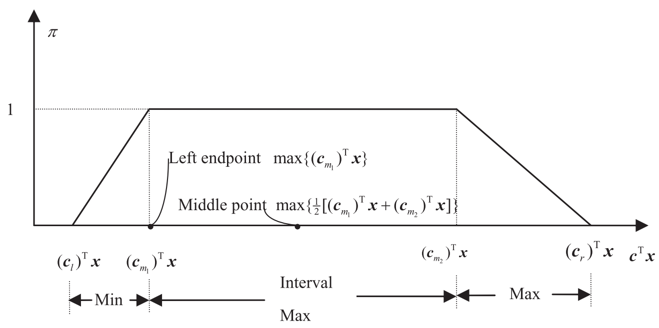

Fig. 1. The illustration for solving $\max \tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}$.

The linear membership function of the objective function $z_i (i = 2, 3, 4)$ can be calculated as follows:

$$\mu_{z_i}(\mathbf{x}) = \begin{cases} 1 & \text{if } z_i > z_i^{\max}, \\ (z_i - z_i^{\min})/(z_i^{\max} - z_i^{\min}) & \text{if } z_i^{\min} \leqslant z_i \leqslant z_i^{\max}, \\ 0 & \text{if } z_i < z_i^{\min}. \end{cases} \tag{7}$$

Thus, Eq. (4) can be solved by the following linear programming model:

$$\begin{aligned} &\max \ \mu, \\ &s.t. \begin{cases} 4\mu_{z_i} + \sum_{i=1}^{4} \mu_{z_i} \geqslant 8\mu \ (i = 1, 2, 3, 4), \\ \mathbf{Ax} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases} \end{aligned} \tag{8}$$

or

$$\begin{aligned} &\max \ \mu, \\ &s.t. \begin{cases} 4\mu_{z_i} + \sum_{i=1}^{4} \mu_{z_i} \leqslant 8\mu \ (i = 1, 2, 3, 4), \\ \mathbf{Ax} \leqslant \mathbf{b} \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases} \end{aligned} \tag{9}$$

or

$$\begin{aligned} &\max \ \left\{ w_1 \mu_{z_1}(\mathbf{x}) + w_2 \mu_{z_2}(\mathbf{x}) + w_3 \mu_{z_3}(\mathbf{x}) + w_4 \mu_{z_4}(\mathbf{x}) \right\}, \\ &s.t. \begin{cases} \mathbf{Ax} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases} \end{aligned} \tag{10}$$

where $\mathbf{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}}$ is the weight vector of objective $z_i \ (i = 1, 2, 3, 4)$, satisfies that $w_i \geqslant 0 \ (i = 1, 2, 3, 4)$ and $\sum_{i=1}^{4} w_i = 1$.

Eq. (8) is a kind of pessimistic approach which shows that the decision maker (DM) is very conservative, whereas Eq. (9) is a kind of optimistic approach which shows that the DM is very aggressive. Eq. (10) is the linear sum approach based on membership function.

It is not difficult to prove that the optimal solutions of Eqs. (8)–(10) are all Pareto optimal solutions of Eq. (4).

3.2. Imprecise objective and technological coefficients $(\tilde{c}, \tilde{A})$

Suppose that in Eq. (1) $\tilde{\mathbf{c}}$ is imprecise objective coefficient, $\tilde{\mathbf{A}}$ is imprecise technological coefficient matrix and $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)^{\mathrm{T}}$ is crisp available resource vector. That is, Eq. (1) can be rewritten as follows:

$$\begin{aligned} &\max \ \{\tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}\}, \\ &s.t. \begin{cases} \tilde{\mathbf{A}}x \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}. \end{cases} \end{aligned} \tag{11}$$

Due to that the constraints of Eq. (11) still contain fuzzy constraints $\tilde{\mathbf{A}}x \leqslant \mathbf{b}$, we propose a weighted average method to deal with these constraints. That is, $(\omega_1 \mathbf{A}_{l\beta} + \omega_2 \mathbf{A}_{m_1\beta} + \omega_3 \mathbf{A}_{m_2\beta} + \omega_4 \mathbf{A}_{r\beta})\mathbf{x} \leqslant \mathbf{b}$, where $\omega_i \geqslant 0 (i = 1, 2, 3, 4)$, $\sum_{i=1}^{4} \omega_i = 1, \beta \in [0, 1]$ is the minimal acceptable possibility, $\mathbf{A}_{l\beta} = \mathbf{A}_l + \beta(\mathbf{A}_{m_1} - \mathbf{A}_l)$, $\mathbf{A}_{m_1\beta} = \mathbf{A}_{m_1} + \beta(\mathbf{A}_{m_1} - \mathbf{A}_{m_1}) = \mathbf{A}_{m_1}$, $\mathbf{A}_{m_2\beta} = \mathbf{A}_{m_2} + \beta(\mathbf{A}_{m_2} - \mathbf{A}_{m_2}) = \mathbf{A}_{m_2}$, $\mathbf{A}_{r\beta} = \mathbf{A}_r - \beta(\mathbf{A}_r - \mathbf{A}_{m_2})$. Thus, the auxiliary model is obtained as follows:

$$\begin{aligned} &\min \ \{z_1 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}\}, \\ &\max \ \{z_2 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}\}, \\ &\max \ \left\{ z_3 = \frac{1}{2}[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} + (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x})] \right\}, \\ &\max \ \{z_4 = (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}\}, \\ &s.t. \begin{cases} \left[\frac{1}{6}(\mathbf{A}_{l\beta} + 2\mathbf{A}_{m_1\beta} + 2\mathbf{A}_{m_2\beta} + \mathbf{A}_{r\beta})\right]\mathbf{x} \leqslant \mathbf{b}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases} \end{aligned} \tag{12}$$

where the weights are $\omega_1 = \frac{1}{6}, \omega_2 = \frac{2}{6}, \omega_3 = \frac{2}{6}, \omega_4 = \frac{1}{6}$. Similar to TrFNs, $\mathbf{A}_{l\beta}$ is relative to the lower limit and is too pessimistic, $\mathbf{A}_{r\beta}$ is relative to the upper limit and is too optimistic, less weights should be assigned to them. However, $\mathbf{A}_{m_1\beta} = \mathbf{A}_{m_1}$ and $\mathbf{A}_{m_2\beta} = \mathbf{A}_{m_2}$ are the most possible values, which can provide the most important and valuable information. Thus, it is very natural to assign more weights to them.

If β is given, Eq. (12) is a multi-objective linear programming model, which can be similarly solved by the method of solving Eq. (4).

3.3. Imprecise objective and technological coefficients and imprecise available resources $(\tilde{c}, \tilde{A}, \tilde{b})$

If $\tilde{\mathbf{c}}, \tilde{\mathbf{A}}$ and $\tilde{\mathbf{b}}$ are all imprecise, then the possibility linear programming with TrFNs is just Eq. (1). On the basis of the strategy for the fuzzy objective function, we combine the fuzzy ranking concepts to handle these fuzzy constraints. Then, the auxiliary model is obtained as follows:

$$
\begin{aligned}
&\min \{z_1 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}\},\\
&\max \{z_2 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}\},\\
&\max \{z_3 = \frac{1}{2}[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} + (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x})]\},\\
&\max \{z_4 = (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}\},\\
&s.t.\ \begin{cases} \mathbf{A}_{l\beta}\mathbf{x} \leqslant \mathbf{b}_{l\beta}, \\ \mathbf{A}_{m_1\beta}\mathbf{x} \leqslant \mathbf{b}_{m_1\beta}, \\ \mathbf{A}_{m_2\beta}\mathbf{x} \leqslant \mathbf{b}_{m_2\beta}, \\ \mathbf{A}_{r\beta}\mathbf{x} \leqslant \mathbf{b}_{r\beta}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases}
\end{aligned}
\tag{13}
$$

where $\mathbf{b}_{l\beta} = \mathbf{b}_l + \beta(\mathbf{b}_{m_1} - \mathbf{b}_l),\ \mathbf{b}_{m_1\beta} = \mathbf{b}_{m_1} + \beta(\mathbf{b}_{m_1} - \mathbf{b}_{m_1}) = \mathbf{b}_{m_1},\ \mathbf{b}_{m_2\beta} = \mathbf{b}_{m_2} + \beta(\mathbf{b}_{m_2} - \mathbf{b}_{m_2}) = \mathbf{b}_{m_2},\ \mathbf{b}_{r\beta} = \mathbf{b}_r - \beta(\mathbf{b}_r - \mathbf{b}_{m_2})$.

Likewise, if β is given, Eq. (13) is a multi-objective linear programming model, which can also be similarly solved by the method of solving Eq. (4).

Obviously, if all TrFNs in the possibility linear programming with TrFNs (i.e., Eq. (1)) are TFNs, that is, all $a_{ijm_1} = a_{ijm_2}, c_{jm_1} = c_{jm_2}$ and $b_{im_1} = b_{im_2} (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, then $\mathbf{A}_{m_1} = \mathbf{A}_{m_2}, \mathbf{c}_{m_1} = \mathbf{c}_{m_2}, \mathbf{A}_{m_1\beta} = \mathbf{A}_{m_2\beta}, \mathbf{b}_{m_1\beta} = \mathbf{b}_{m_2\beta}$. Then, Eq. (13) is reduced to the following auxiliary model:

$$
\begin{aligned}
&\min \{z_1 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}\},\\
&\max \{z_2 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}\},\\
&\max \{z_4 = (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x}\},\\
&s.t.\ \begin{cases} \mathbf{A}_{l\beta}\mathbf{x} \leqslant \mathbf{b}_{l\beta}, \\ \mathbf{A}_{m_1\beta}\mathbf{x} \leqslant \mathbf{b}_{m_1\beta}, \\ \mathbf{A}_{r\beta}\mathbf{x} \leqslant \mathbf{b}_{r\beta}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases}
\end{aligned}
$$

which is just the same as the auxiliary model (i.e., Eq. (13)) obtained in [2]. In other words, the possibility linear programming with TrFNs of this paper is reduced to the possibility linear programming with TFNs of [2]. Therefore, the former is the extension of the latter indeed.

Analogously, for a minimization possibility linear programming with TrFNs, i.e.,

$$
\begin{aligned}
&\min \{\tilde{z} = \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{x}\},\\
&s.t.\ \begin{cases} \tilde{\mathbf{A}}x \leqslant \tilde{\mathbf{b}}, \\ \mathbf{x} \geqslant \mathbf{0}, \end{cases}
\end{aligned}
\tag{14}
$$

using Definition 2, we have the following equivalent auxiliary model:

$$
\begin{aligned}
&\max \{z_1 = (\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_l)^{\mathrm{T}}\mathbf{x}\},\\
&\min \left\{z_2 = (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}\right\},\\
&\min \{z_3 = \frac{1}{2}[(\mathbf{c}_{m_1})^{\mathrm{T}}\mathbf{x} + (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x})]\},\\
&\min \{z_4 = (\mathbf{c}_r)^{\mathrm{T}}\mathbf{x} - (\mathbf{c}_{m_2})^{\mathrm{T}}\mathbf{x}\},\\
&s.t.\ \begin{cases} \mathbf{A}_{l\beta}\mathbf{x} \leqslant \mathbf{b}_{l\beta}, \\ \mathbf{A}_{m_1\beta}\mathbf{x} \leqslant \mathbf{b}_{m_1\beta}, \\ \mathbf{A}_{m_2\beta}\mathbf{x} \leqslant \mathbf{b}_{m_2\beta}, \\ \mathbf{A}_{r\beta}\mathbf{x} \leqslant \mathbf{b}_{r\beta}, \\ \mathbf{x} \geqslant \mathbf{0}. \end{cases}
\end{aligned}
\tag{15}
$$

Eq. (15) can be solved by using the methods similar to Eqs. (5)–(10).

4. A real investment problem and comparison analysis of the obtained results

4.1. An investment problem and the analysis process

The proposed method is applied to an investment problem adapted from [2]. Winston-Salem Development Management (WSDM) is trying to complete its investment plans for the next 3 years. Taking 6-month interval as a period, WSDM gives the expectations of the income streams (in millions) for the next 3 years, shown in Table 1. In this particular case, \$0.3 million at the end of third year will not play any important role.

WSDM desires to participate in three development projects. Project 1 is the Foster City Development. Project 2 is to take over the operation of some old, lower-middle-income housing on the condition that certain initial repairs to it be made and that it be demolished at the end of 3 years. Project 3 is to invest the Disney-Universe Hotel. If WSDM fully participates in these three projects, it would have the projected cash-flow stream (in millions) at 6-month intervals over the next 3 years as shown in Table 1 where the negative numbers represent investments, positive numbers represent income.

Assume that currently WSDM has \$2 million available for investment and can borrow money for half-year intervals at 6% interest per half-year. At most, \$2 million can be borrowed at one time; that is, the outstanding principal can never exceed \$2 million. WSDM can invest surplus funds at 4% per half-year.

Case 1. Imprecise cash-flow streams.

Suppose that the cash-flow streams for the 6th period are imprecise and represented by TrFNs. For example, the cash-flow stream for Project 1 is a TrFN $(5.0, 5.2, 5.5, 6.2)$, which means that the most possible value of this cash-flow stream is between $[5.2, 5.5]$, the lower and upper limits are 5.0 and 6.2, respectively. If WSDM participates in a project at less than 100%, all the cash flows of that project are reduced proportionately.

The investment object of WSDM is to maximize the net worth at the end of 3 years. Then, the problem considered is how to reasonably allocate the participation proportions of each project, and the amounts borrowed and lent in each period. Therefore, we first introduce the following decision variables:

F = fractional participation in Foster City.
M = fractional participation in Lower–Middle.
D = fractional participation in Disney-Universe.
B_i = amount borrowed in period $i (i = 1, 2, \ldots, 6)$.
L_i = amount lent in period $i (i = 1, 2, \ldots, 6)$.
$\tilde{z}$ = net worth after the six periods (without considering \$0.3 million).

The fuzzy linear programming model is set up as follows:

$$\max\ \{\tilde{z} = (5.0, 5.2, 5.5, 6.2)F + (-1.4, -1.2, -1.0, -0.85)M + (4.5, 5.0, 6.0, 6.5)D - 1.06B_6 + 1.04L_6\},$$
$$s.t.\ \begin{cases} 3F + 2M + 2D - B_1 + L_1 \leqslant 2, \\ F + 0.5M + 2D + 1.06B_1 - 1.04L_1 - B_2 + L_2 \leqslant 0.5, \\ 1.8F - 1.5M + 1.8D + 1.06B_2 - 1.04L_2 - B_3 + L_3 \leqslant 0.4, \\ -0.4F - 1.5M - D + 1.06B_3 - 1.04L_3 - B_4 + L_4 \leqslant 0.38, \\ -1.8F - 1.5M - D + 1.06B_4 - 1.04L_4 - B_5 + L_5 \leqslant 0.36, \\ -1.8F - 0.2M - D + 1.06B_5 - 1.04L_5 - B_6 + L_6 \leqslant 0.34, \\ 0 \leqslant B_i \leqslant 2\ (i = 1, 2, \ldots, 6), \\ 0 \leqslant F \leqslant 2, 0 \leqslant M \leqslant 2, 0 \leqslant D \leqslant 2. \end{cases} \tag{16}$$

By Eq. (4), Eq. (16) can be solved by the auxiliary multi-objective programming model:

$$\min\ \{z_1 = 0.2F + 0.2M + 0.5D\},$$
$$\max\ \{z_2 = 5.2F - 1.2M + 5.0D - 1.06B_6 + 1.04L_6\},$$
$$\max\ \{z_3 = 5.35F - 1.1M + 5.5D - 1.06B_6 + 1.04L_6\},$$
$$\max\ \{z_4 = 0.7F + 0.15M + 0.5D\},$$
$$s.t.\ \begin{cases} 3F + 2M + 2D - B_1 + L_1 \leqslant 2, \\ F + 0.5M + 2D + 1.06B_1 - 1.04L_1 - B_2 + L_2 \leqslant 0.5, \\ 1.8F - 1.5M + 1.8D + 1.06B_2 - 1.04L_2 - B_3 + L_3 \leqslant 0.4, \\ -0.4F - 1.5M - D + 1.06B_3 - 1.04L_3 - B_4 + L_4 \leqslant 0.38, \\ -1.8F - 1.5M - D + 1.06B_4 - 1.04L_4 - B_5 + L_5 \leqslant 0.36, \\ -1.8F - 0.2M - D + 1.06B_5 - 1.04L_5 - B_6 + L_6 \leqslant 0.34, \\ 0 \leqslant B_i \leqslant 2\ (i = 1, 2, \ldots, 6), \\ 0 \leqslant F \leqslant 2, 0 \leqslant M \leqslant 2, 0 \leqslant D \leqslant 2. \end{cases} \tag{17}$$

Table 1
The multiple project and multiple period investment problem (in million dollars).

Period	0	1	2	3	4	5	6
Year	0	0.5	1	1.5	2	2.5	3
Income stream	2.0	0.5	0.4	0.38	0.36	0.34	0.3
Project 1	−3.0	−1.0	−1.8	0.4	1.8	1.8	(5.0, 5.2, 5.5, 6.2)
Project 2	−2.0	−0.5	1.5	1.5	1.5	0.2	(−1.4, −1.2, −1.0, −0.85)
Project 3	−2.0	−2.0	−1.8	1.0	1.0	1.0	(4.5, 5.0, 6.0, 6.5)

Using Eq. (5), it follows that

$$z_1^{\max} = 0.4831, \quad z_2^{\max} = 6.8880, \quad z_3^{\max} = 7.0576, \quad z_4^{\max} = 0.5859,$$

$$z_1^{\min} = 0.0000, \quad z_2^{\min} = -3.3200, \quad z_3^{\min} = -3.3200, \quad z_4^{\min} = 0.0000.$$

Next, we use three approaches (i.e., Eqs. (8)–(10)) to solving Eq. (17), respectively.

First, solving Eq. (17) by using Eq. (8) (i.e., pessimistic approach), we obtain the optimal solution as follows:

$$F = 0.34, \quad M = 0.67, \quad D = 0, \quad B_1 = 0.37, \quad B_2 = 0.57, \quad B_3 = 0, \quad B_4 = 0, \quad B_5 = 0, \quad B_6 = 0,$$

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 0.18, \quad L_4 = 1.71, \quad L_5 = 3.76, \quad L_6 = 5.$$

Substituted the above optimal solution into the objective function of Eq. (16), the optimal objective value is obtained as (5.98, 6.19, 6.42, 6.76). Namely, the most likely profit is between 6.19 and 6.42, the upper and lower limits of the profits of the problem are 6.76 and 5.98, respectively.

Second, solving Eq. (17) by using Eq. (9) (i.e., optimistic approach), we obtain the optimal solution as follows:

$$F = 1, \quad M = 0, \quad D = 0, \quad B_1 = 0, \quad B_2 = 0, \quad B_3 = 2, \quad B_4 = 2, \quad B_5 = 2, \quad B_6 = 2,$$

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = 0, \quad L_5 = 2, \quad L_6 = 0.$$

Substituted the above optimal solution into the objective function of Eq. (16), the optimal objective value is obtained as (−3.52, −3.32, −3.12, −2.97). Namely, the most likely profit is between −3.32 and −3.12, the upper and lower limits of the profits of the problem are −2.97 and −3.52, respectively. This result is accordance with the intuition, because the DM is very optimistic and aggressive, which results in the negative profit.

Finally, solving Eq. (17) by using Eq. (10) (i.e., linear sum approach based on membership function), we obtain the optimal solution for $\mathbf{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^{\mathrm{T}}$ as follows:

$$F = 0.83, \quad M = 0, \quad D = 0, \quad B_1 = 0.5, \quad B_2 = 0.85, \quad B_3 = 2, \quad B_4 = 1.41, \quad B_5 = 0, \quad B_6 = 0,$$

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = 0, \quad L_5 = 0.36, \quad L_6 = 2.21.$$

Replaced the above optimal solution into the objective function of Eq. (16), the optimal objective value is obtained as (6.46, 6.62, 6.87, 7.45). Namely, the most likely profit is between 6.62 and 6.87, the upper and lower limits of the profits of the problem are 7.45 and 6.46, respectively.

To illustrate the influence of the weight vector $\mathbf{w}$ on the optimal profit in this example, we use a different weight vector $\mathbf{w}$ to solve Eq. (17) according to Eq. (10). Generally speaking, we need to consider some special cases. One is an average weight, i.e, $\mathbf{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^{\mathrm{T}}$. Secondly, the objective functions z_2 and z_3 are relative to the mode of TrFN $\tilde{z}$, which are the most possible values of $\tilde{z}$. Thus, more weights should be assigned to z_2 and z_3, i.e., $\mathbf{w} = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^{\mathrm{T}}$. Thirdly, similar to the Olympic games, which discarded a maximum point and a minimum point, we can set $\mathbf{w} = (0, \frac{1}{2}, \frac{1}{2}, 0)^{\mathrm{T}}$. Finally, $\mathbf{w} = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^{\mathrm{T}}$ emphasizes both ends and reduces the middle. All the computation results are shown in Table 2.

It can be seen from Table 2 that, the optimal solutions and optimal profits for $\mathbf{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^{\mathrm{T}}, \mathbf{w} = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^{\mathrm{T}}$ and $\mathbf{w} = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^{\mathrm{T}}$ are completely identical. The optimal solution and optimal profit for $\mathbf{w} = (0, \frac{1}{2}, \frac{1}{2}, 0)^{\mathrm{T}}$ are not the same as that for the other weight vectors. However this does not signify that the weight vector $\mathbf{w}$ does not affect the optimal solutions and optimal profits. The case 2 will illustrate this effect.

In addition, Table 2 shows that applying different approaches to solving the multi-objective programming may result in different optimal solutions and optimal objective values. The DM can choose the proper approach to solving the multi-objective programming according to his/her risk preference and actual requirements.

Case 2. Imprecise cash-flow streams and interest rates.

Since the financial market is very complex and constantly changing, the interest rate is not fixed and may vary at different periods. It is appropriate for DM to use TrFNs to express the fuzziness and uncertainties inherent in the interest rate. For instance, the floating interest rate is represented as TrFN (6%, 8%, 9%, 11%), which implies that this interest rate will be

Table 2
The optimal solution and optimal profit for case 1 with different approaches.

Variable	F	M	D	B_1	B_2	B_3	B_4	B_5	B_6	L_1	L_2	L_3	L_4	L_5	L_6	Optimal profit
By Eq. (8)	0.34	0.67	0	0.37	0.57	0	0	0	0	0	0	0.18	1.71	3.76	5	(5.98, 6.19, 6.42, 6.76)
By Eq. (9)	1	0	0	0	0	2	2	2	2	0	0	0	0	2	0	(−3.52, −3.32, −3.12, −2.97)
By Eq. (10) with **w**																
$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})^{\mathrm{T}}$	0.83	0	0	0.5	0.85	2	1.41	0	0	0	0	0	0	0.36	2.21	(6.46, 6.62, 6.87, 7.45)
$(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6})^{\mathrm{T}}$	0.83	0	0	0.5	0.85	2	1.41	0	0	0	0	0	0	0.36	2.21	(6.46, 6.62, 6.87, 7.45)
$(0,\frac{1}{2},\frac{1}{2},0)^{\mathrm{T}}$	0.7	0.65	0	1.39	2	2	0.49	0	0	0	0	0	0	2.08	3.89	(6.62, 6.89, 7.23, 7.81)
$(\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{2}{6})^{\mathrm{T}}$	0.83	0	0	0.5	0.85	2	1.41	0	0	0	0	0	0	0.36	2.21	(6.46, 6.62, 6.87, 7.45)

between 8% and 9%, not less that 6% and not larger than 11% in the next years. Hence, it is necessary to study the case where interest rate has a trapezoidal possibility distribution.

In what follows, we consider the floating interest rate, the coefficients of the objective and the third through the sixth constraints of Eq. (16) as imprecise or fuzzy. The first and second constraints are the beginning of the planning periods which are quite sure. However, the interest rate will fluctuate in the future year. Then, considering the imprecise cash-flow streams and interest rates simultaneously, we may formulate the following fuzzy linear programming with TrFNs:

$$\max\ \{\tilde{z} = (5.0,5.2,5.5,6.2)F + (-1.4,-1.2,-1.0,-0.85)M + (4.5,5.0,6.0,6.5)D - 1.06B_6 + 1.04L_6\},$$
$$s.t.\ \begin{cases} 3F + 2M + 2D - B_1 + L_1 \leqslant 2, \\ F + 0.5M + 2D + 1.06B_1 - 1.04L_1 - B_2 + L_2 \leqslant 0.5, \\ 1.8F - 1.5M + 1.8D + (1.050,1.055,1.06,1.065)B_2 - (1.030,1.035,1.04,1.045)L_2 - B_3 + L_3 \leqslant 0.4, \\ -0.4F - 1.5M - D + (1.050,1.055,1.06,1.07)B_3 - (1.030,1.035,1.04,1.05)L_3 - B_4 + L_4 \leqslant 0.38, \\ -1.8F - 1.5M - D + (1.05,1.06,1.065,1.07)B_4 - (1.035,1.038,1.044,1.05)L_4 - B_5 + L_5 \leqslant 0.36, \\ -1.8F - 0.2M - D + (1.055,1.058,1.065,1.075)B_5 - (1.040,1.042,1.046,1.055)L_5 - B_6 + L_6 \leqslant 0.34, \\ 0 \leqslant B_i \leqslant 2\ (i = 1,2,\ldots,6), \\ 0 \leqslant F \leqslant 2, 0 \leqslant M \leqslant 2, 0 \leqslant D \leqslant 2. \end{cases} \tag{18}$$

By Eq. (12), Eq. (18) can be solved by the auxiliary multi-objective programming model with $\beta = 0.5$:

$$\begin{aligned} &\min\ \{z_1 = 0.2F + 0.2M + 0.5D\}, \\ &\max\ \{z_2 = 5.2F - 1.2M + 5.0D - 1.06B_6 + 1.04L_6\}, \\ &\max\ \{z_3 = 5.35F - 1.1M + 5.5D - 1.06B_6 + 1.04L_6\}, \\ &\max\ \{z_4 = 0.7F + 0.15M + 0.5D\}, \\ &s.t.\ \begin{cases} 3F + 2M + 2D - B_1 + L_1 \leqslant 2, \\ F + 0.5M + 2D + 1.06B_1 - 1.04L_1 - B_2 + L_2 \leqslant 0.5, \\ 1.8F - 1.5M + 1.8D + 1.0575B_2 - 1.0383L_2 - B_3 + L_3 \leqslant 0.4, \\ -0.4F - 1.5M - D + 1.0579B_3 - 1.0379L_3 - B_4 + L_4 \leqslant 0.38, \\ -1.8F - 1.5M - D + 1.0621B_4 - 1.0413L_4 - B_5 + L_5 \leqslant 0.36, \\ -1.8F - 0.2M - D + 1.0621B_5 - 1.0446L_5 - B_6 + L_6 \leqslant 0.34, \\ 0 \leqslant B_i \leqslant 2\ (i = 1,2,\ldots,6), \\ 0 \leqslant F \leqslant 2, 0 \leqslant M \leqslant 2, 0 \leqslant D \leqslant 2. \end{cases} \end{aligned} \tag{19}$$

Analogously, we use three approaches (i.e., Eqs. (8)–(10)) to solving Eq. (19), respectively. The obtained results are listed in Table 3.

Table 3 shows that by using the pessimistic approach (i.e., Eq. (8)), the optimal objective value is (6.01, 6.21, 6.45, 6.79). Namely, the upper and lower of the profits of the problem are 6.79 and 6.01, the most likely profit is between 6.21 and 6.45. By using the optimistic approach (i.e., Eq. (9)), the optimal objective value is (−3.52, −3.32, −3.12, −2.97). Namely, the upper and lower of the profits of the problem are −2.97 and −3.52, the most likely profit is between −3.32 and −3.12.

By using the linear sum method based on membership function (i.e., Eq. (10)), the optimal solutions and optimal profits for $\mathbf{w} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})^{\mathrm{T}}$ and $\mathbf{w} = (\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6})^{\mathrm{T}}$ are completely identical. The optimal solution and optimal profit for $\mathbf{w} = (0,\frac{1}{2},\frac{1}{2},0)^{\mathrm{T}}$ and $\mathbf{w} = (\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{2}{6})^{\mathrm{T}}$ are not the same as that for $\mathbf{w} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})^{\mathrm{T}}$ and $\mathbf{w} = (\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6})^{\mathrm{T}}$. This signifies that the weight vector $\mathbf{w}$ does affect the optimal solutions and optimal profits.

Case 3. Imprecise cash-flow streams, interest rates, and income streams.

With ever increasing complexity in real finical word, there are often some challenges for the DM to provide precise information of income stream due to time pressure, lack of knowledge (or data) and the DM's limited expertise about the problem domain. The income stream may not be always precise as time progresses. Some of them may be expressed with TrFNs.

Table 3
The optimal solution and optimal objective value for case 2 with different approaches.

Variable	F	M	D	B_1	B_2	B_3	B_4	B_5	B_6	L_1	L_2	L_3	L_4	L_5	L_6	Optimal profit
By Eq. (8)	0.34	0.67	0	0.37	0.57	0	0	0	0	0	0	0.18	1.71	3.76	5.02	(6.01, 6.21, 6.45, 6.79)
By Eq. (9)	0	1	0	0	0	2	2	2	2	0	0	0	0	0	0	(−3.52, −3.32, −3.12, −2.97)
By Eq. (10) with **w**																
$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})^{\mathrm{T}}$	0.83	0	0	0.5	0.85	2	1.40	0	0	0	0	0	0	0.37	2.22	(6.46, 6.63, 6.88, 7.46)
$(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6})^{\mathrm{T}}$	0.83	0	0	0.5	0.85	2	1.40	0	0	0	0	0	0	0.37	2.22	(6.46, 6.63, 6.88, 7.46)
$(0,\frac{1}{2},\frac{1}{2},0)^{\mathrm{T}}$	0.7	0.65	0	1.39	2	2	0.49	0	0	0	0	0	0	2.08	3.89	(6.64, 6.91, 7.25, 7.83)
$(\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{2}{6})^{\mathrm{T}}$	0.83	0	0	0.5	0.86	2	1.4	0	0	0	0	0	0	0.37	2.22	(6.64, 6.63, 6.88, 7.46)

For the above investment problems, we modify the "the expectations of income stream" of Table 1 as: 0.5, (0.30, 0.35, 0.40, 0.50), (0.30, 0.35, 0.38, 0.50), (0.28, 0.34, 0.36, 0.45) and (0.25, 0.30, 0.34, 0.42), respectively. The other assumptions are the same as the previous ones. Then, considering the imprecise cash-flow streams, interest rates, and income streams simultaneously, we may construct the fuzzy linear programming with TrFNs as follows:

$$\max\ \{\tilde{z} = (5.0, 5.2, 5.5, 6.2)F + (-1.4, -1.2, -1.0, -0.85)M + (4.5, 5.0, 6.0, 6.5)D - 1.06B_6 + 1.04L_6\},$$
$$s.t.\ \begin{cases} 3F + 2M + 2D - B_1 + L_1 \leqslant 2, \\ F + 0.5M + 2D + 1.06B_1 - 1.04L_1 - B_2 + L_2 \leqslant 0.5, \\ 1.8F - 1.5M + 1.8D + (1.050, 1.055, 1.06, 1.065)B_2 - (1.030, 1.035, 1.04, 1.045)L_2 - B_3 + L_3 \leqslant (0.30, 0.35, 0.40, 0.50), \\ -0.4F - 1.5M - D + (1.050, 1.055, 1.06, 1.07)B_3 - (1.030, 1.035, 1.04, 1.05)L_3 - B_4 + L_4 \leqslant (0.30, 0.35, 0.38, 0.40), \\ -1.8F - 1.5M - D + (1.05, 1.06, 1.065, 1.07)B_4 - (1.035, 1.038, 1.044, 1.05)L_4 - B_5 + L_5 \leqslant (0.28, 0.34, 0.36, 0.45), \\ -1.8F - 0.2M - D + (1.055, 1.058, 1.065, 1.075)B_5 - (1.040, 1.042, 1.046, 1.055)L_5 - B_6 + L_6 \leqslant (0.25, 0.30, 0.34, 0.42), \\ 0 \leqslant B_i \leqslant 2\ (i = 1, 2, \ldots, 6), \\ 0 \leqslant F \leqslant 2, 0 \leqslant M \leqslant 2, 0 \leqslant D \leqslant 2. \end{cases} \tag{20}$$

By Eq. (13), Eq. (20) can be solved by the auxiliary multi-objective programming model with $\beta = 0.5$:

$$\begin{aligned} &\min\ \{z_1 = 0.2F + 0.2M + 0.5D\}, \\ &\max\ \{z_2 = 5.2F - 1.2M + 5.0D - 1.06B_6 + 1.04L_6\}, \\ &\max\ \{z_3 = 5.35F - 1.1M + 5.5D - 1.06B_6 + 1.04L_6\}, \\ &\max\ \{z_4 = 0.7F + 0.15M + 0.5D\}, \end{aligned}$$
$$s.t.\ \begin{cases} 3F + 2M + 2D - B_1 + L_1 \leqslant 2, \\ F + 0.5M + 2D + 1.06B_1 - 1.04L_1 - B_2 + L_2 \leqslant 0.5, \\ 1.8F - 1.5M + 1.8D + 1.0525B_2 - 1.0325L_2 - B_3 + L_3 \leqslant 0.3250, \\ 1.8F - 1.5M + 1.8D + 1.055B_2 - 1.035L_2 - B_3 + L_3 \leqslant 0.35, \\ 1.8F - 1.5M + 1.8D + 1.06B_2 - 1.035L_2 - B_3 + L_3 \leqslant 0.4, \\ 1.8F - 1.5M + 1.8D + 1.06255B_2 - 1.0475L_2 - B_3 + L_3 \leqslant 0.4500, \\ -0.4F - 1.5M - D + 1.0525B_3 - 1.0325L_3 - B_4 + L_4 \leqslant 0.3250, \\ -0.4F - 1.5M - D + 1.055B_3 - 1.035L_3 - B_4 + L_4 \leqslant 0.35, \\ -0.4F - 1.5M - D + 1.06B_3 - 1.04L_3 - B_4 + L_4 \leqslant 0.38, \\ -0.4F - 1.5M - D + 1.0650B_3 - 1.0450L_3 - B_4 + L_4 \leqslant 0.3900, \\ -1.8F - 1.5M - D + 1.0550B_4 - 1.0365L_4 - B_5 + L_5 \leqslant 0.3100, \\ -1.8F - 1.5M - D + 1.06B_4 - 1.038L_4 - B_5 + L_5 \leqslant 0.34, \\ -1.8F - 1.5M - D + 1.065B_4 - 1.044L_4 - B_5 + L_5 \leqslant 0.36, \\ -1.8F - 1.5M - D + 1.0675B_4 - 1.0470L_4 - B_5 + L_5 \leqslant 0.4050, \\ -1.8F - 0.2M - D + 1.0565B_5 - 1.0410L_5 - B_6 + L_6 \leqslant 0.2750, \\ -1.8F - 0.2M - D + 1.058B_5 - 1.042L_5 - B_6 + L_6 \leqslant 0.30, \\ -1.8F - 0.2M - D + 1.065B_5 - 1.046L_5 - B_6 + L_6 \leqslant 0.34, \\ -1.8F - 0.2M - D + 1.0700B_5 - 1.0505L_5 - B_6 + L_6 \leqslant 0.3800, \\ 0 \leqslant B_i \leqslant 2\ (i = 1, 2, \ldots, 6), \\ 0 \leqslant F \leqslant 2, 0 \leqslant M \leqslant 2, 0 \leqslant D \leqslant 2. \end{cases} \tag{21}$$

Analogously, we use three approaches (i.e., Eqs. (8)–(10)) to solving Eq. (21), respectively. The obtained results are listed in Table 4.

The results for Table 4 can be analyzed similar to that for Table 3.

4.2. Comparison analysis of the obtained results

In this subsection, we compare the results obtained by the method [2] and the proposed method in this paper.

Lai and Hwang [2] assumed that the fuzzy numbers in the above investment problem are all TFNs. They obtained the corresponding optimal profits for the three cases as (7.91, 8.20, 8.72) for case 1, (8.12, 8.41, 8.99) for case 2, and (6.58, 6.86, 7.44) for case 3, respectively. In this paper, we use the pessimistic approach to obtaining the optimal profits for the three cases as (5.98, 6.19, 6.42, 6.76) for case 1, (6.01, 6.21, 6.45, 6.79) for case 2, and (5.70, 5.90, 6.14, 6.48) for case 3, respectively. The comparison for these results is depicted as in Figs. 2–4. For the obtained results by the optimistic approach and the linear sum approach based on membership function, the comparison analysis can also be conducted similarly.

It is easily seen from Figs. 2–4 that the optimal profits for the three cases in [2] and this paper are remarkably different. The optimal profits obtained in the former are TFNs, while that in the latter are TrFNs. The most possible value for TFN is a single real number, whereas that for TrFN is an interval. For example, for case 1 the most possible value obtained in [2] is 8.20, while that in this paper is between 6.19 and 6.42, i.e., interval [6.19, 6.42]. This indicates that TrFNs can express more uncertain information than TFNs.

Moreover, Lai and Hwang [2] only proposed an approach to solving the constructed multi-objective programming, while this paper proposed three different approaches to solving the multi-objective programming. In this paper, different DM can select different approach according to his/her risk preference and actual needs, which greatly enhances the flexibility in the process of decision making.

Since TFNs can be written as TrFNs, if all the TFNs in [2] are written as TrFNs, then the possibility linear programming with TFNs developed in [2] is turned to the possibility linear programming with TrFNs developed in this paper. That is to say, the possibility linear programming in [2] is just a special case of that in this paper.

Compared with [5–7,9,10], the proposed method in this paper is more reliable and stable since it can avoid selecting ranking methods of TrFNs. Furthermore, for different minimal acceptable possibility $\beta \in [0, 1]$, we can obtain the different optimal solution and optimal objective value, which can help the DM to make decision flexibly.

Table 4
The optimal solution and optimal objective value for case 3 with different approaches.

Variable	F	M	D	B_1	B_2	B_3	B_4	B_5	B_6	L_1	L_2	L_3	L_4	L_5	L_6	Optimal profit
By Eq. (8)	0.34	0.67	0	0.35	0.54	0	0	0	0	0	0	0.15	1.61	3.59	4.75	(5.70, 5.90, 6.14, 6.48)
By Eq. (9)	0.58	0.85	0	1.42	2	1.54	0	0	0	0	0	0	0	2.09	3.66	(5.48, 5.76, 6.10, 6.63)
By Eq. (10) with **w**																
$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})^T$	0.82	0	0	0.46	0.81	2	1.45	0	0	0	0	0	0	0.25	2.01	(6.19, 6.36, 6.60, 7.18)
$(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6})^T$	0.82	0	0	0.46	0.81	2	1.45	0	0	0	0	0	0	0.25	2.01	(6.19, 6.36, 6.60, 7.18)
$(0,\frac{1}{2},\frac{1}{2},0)^T$	0.68	0.67	0	1.4	2	2	0.5	0	0	0	0	0	0	2.03	3.75	(6.37, 6.64, 6.98, 7.56)
$(\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{2}{6})^T$	0.82	0	0	0.5	0.81	2	1.45	0	0	0	0	0	0	0.25	2.01	(6.20, 6.36, 6.60, 7.18)

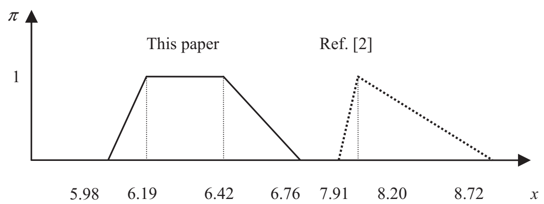

Fig. 2. Optimal profits for case 1.

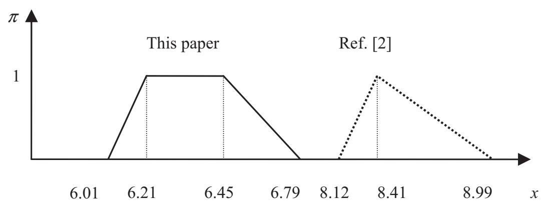

Fig. 3. Optimal profits for case 2.

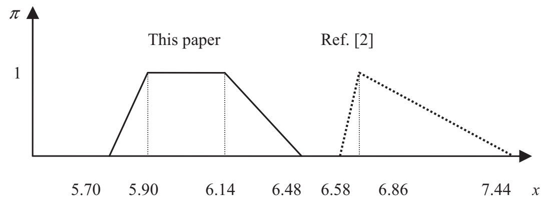

Fig. 4. Optimal profits for case 3.

Table 5
The unit cost, capacity and demand for a transportation problem.

	Market 1	Market 2	Market 3	Capacity
Port 1	(7,8,9,10)	(5,6,7,8)	(3,4,5,6)	400
Port 2	(1,2,2.5,3)	(5,5.5,6.5,7)	(8,9,9.5,10)	300
Port 3	(6,7,7.5,8.1)	(3,4,4.8,5.4)	(2,2.8,3.5,4.2)	200
Demand	220	450	230	

5. Application to a transportation problem

To further illustrate the potential application of the proposed method in this paper, let us consider a transportation problem in this section. Suppose that there are three ports and three markets. We must load the products from these three ports to three markets. The unit costs of a delivery from the ith port to the jth market (in dollars) are given in the respective cells of Table 5. Due to the complexity of the objective things and the fuzziness of human thoughts, it is not easy for the DM to give the crisp numerical values for evaluating the unit costs. TrFNs are suitable to express the unit costs. The demands of loads in each market and the capacity of loads in each port are given (in kilograms) in the last column (row) of Table 5. An optimal transportation plan which minimizes the total costs has to be determined.

Let the transportation quantity from the ith port to the jth market be denoted by $x_{ij}(i=1,2,3; j=1,2,3)$. The fuzzy linear programming with TrFNs is formulated as follows:

$$\begin{aligned} \min\ \tilde{z} = {} & (7,8,9,10)x_{11} + (5,6,7,8)x_{12} + (3,4,5,6)x_{13} + (1,2,2.5,3)x_{21} + (5,5.5,6.5,7)x_{22} \\ & + (8,9,9.5,10)x_{23} + (6,7,7.5,8.1)x_{31} + (3,4,4.8,5.4)x_{32} + (2,2.8,3.5,4.2)x_{33}, \\ s.t. & \begin{cases} x_{11}+x_{12}+x_{13} \leqslant 400, \\ x_{21}+x_{22}+x_{23} \leqslant 300, \\ x_{31}+x_{32}+x_{33} \leqslant 200, \\ x_{11}+x_{21}+x_{31} \leqslant 220, \\ x_{12}+x_{22}+x_{32} \leqslant 450, \\ x_{13}+x_{23}+x_{33} \leqslant 230, \\ x_{ij} \geqslant 0\ (i=1,2,3;\ j=1,2,3). \end{cases} \end{aligned} \tag{22}$$

According to Eqs. (14) and (15), we adopt the pessimistic approach (i.e., Eq. (8)) to solving Eq. (22). The optimal solution can be obtained as follows:

$$x_{11}=0, x_{12}=176.8806, \quad x_{13}=150, \quad x_{21}=220, \quad x_{22}=0, \quad x_{23}=80, \quad x_{31}=0, \quad x_{32}=200, \quad x_{33}=0.$$

Therefore, the total cost is a TrFN $\tilde{z}$ = (2794.4,3621.3,4258.2,4855.0), which shows that the most possible total cost is between [3621.3,4258.2], the upper and lower limits of the total cost are 4855.0 and 2794.4, respectively.

6. Conclusions

This paper developed a new possibility linear programming with TrFNs. For the imprecise objective coefficients and/or the imprecise technological coefficients and/or resources, we proposed the auxiliary multi-objective programming to solve the corresponding possibility linear programming with TrFNs. The auxiliary multi-objective programming involved four objectives: minimizing the left spread, maximizing the right spread, maximizing the left endpoint of the mode and maximizing the middle point of the mode.

The comparison analysis of the investment problem shows that the proposed possibility linear programming with TrFNs generalizes the possibility linear programming with TFNs [2]. Furthermore, this paper proposed three kinds of approaches to solving the constructed auxiliary multi-objective programming. Different DM can select different approach according to his/her risk preference and actual needs, which greatly enhances the flexibility in the process of decision making. Although the developed method was illustrated using an investment problem and a transportation problem, it will be expected to be applicable to real-life decision problems in many areas, such as risk investment, engineering management, supply chain management.

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