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Thermal buckling and post-buckling of pinned–fixed Euler–Bernoulli beams on an elastic foundation

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Abstract

In this article, both thermal buckling and post-buckling of pinned–fixed beams resting on an elastic foundation are investigated. Based on the accurate geometrically non-linear theory for Euler–Bernoulli beams, considering both linear and non-linear elastic foundation effects, governing equations for large static deformations of the beam subjected to uniform temperature rise are derived. Due to the large deformation of the beam, the constraint forces of elastic foundation in both longitudinal and transverse directions are taken into account. The boundary value problem for the non-linear ordinary differential equations is solved effectively by using the shooting method. Characteristic curves of critical buckling temperature versus elastic foundation stiffness parameter corresponding to the first, the second, and the third buckling mode shapes are plotted. From the numerical results it can be found that the buckling load-elastic foundation stiffness curves have no intersection when the value of linear foundation stiffness parameter is less than 3000, which is different from the behaviors of symmetrically supported (pinned–pinned and fixed–fixed) beams. As we expect that the non-linear foundation stiffness parameter has no sharp influence on the critical buckling temperature and it has a slight effect on the post-buckling temperature compared with the linear one.

Keywords: Beam; Elastic foundation; Thermal buckling; Mode transition; Numerical results

1. Introduction

Thermal buckling may be an undesired phenomenon in many structures such as railroad tracks, pipelines, and concrete roads. Some cases cannot be avoided under special conditions. So, in recent years, many researchers have paid close attention to finding the regularity of thermal buckling to ensure the safety of structures. A number of papers on thermal buckling of beams have been published in recent years. Jekot (1996) examined the thermal post-buckling of a beam made of physically non-linear thermo-elastic material, in which he considered a simplified form of axial strain rather than the geometric non-linearity of the curvature of deformed central axis. By accurately considering the formulation of the axial strain and the curvature, Coffin and Bloom (1999) first presented an elliptic integral solution for the symmetric post-buckling response of a

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linear-elastic and hygrothermal rod with two ends pinned. However, a numerical solution to two coupled elliptical equations is necessary for the final post-buckling solution. Assuming that thermal strain temperature is non-linear, Vaz and Solano (2003a,b) also examined thermal post-buckling of rods and came up with a closed-form solution via uncoupled elliptical integrals. But, due to the limit of the elliptical integral to the boundary conditions, only the case of pinned–pinned ends was considered. In the light of the exact non-linear geometric theory, Li and Cheng (2000), Li et al. (2002) and Li and Xi (2006) presented accurate mathematical formulations for post-buckling of Euler–Bernoulli beams and Timoshenko beams with different boundary conditions. When a static increasing temperature was applied the strongly non-linear differential equations with various boundary conditions were solved numerically by using a shooting method. The strongly non-linear differential equations with various boundary conditions were solved numerically by using the shooting method. Raju and Rao (1993), Rao and Raju (2002) and Rao and Neetha (2002) did a series of investigations on thermal post-buckling of uniform columns as well as tapered columns by Raleigh–Ritz method, finite element method and intuitive method. The effects of elastic foundation parameter on the critical temperature and post-buckling temperature rise were also considered, but they did not take into account the non-linearity of the curvature of the deformed central axis.

In the present paper, both thermal buckling and post-buckling of a pinned–fixed beam resting on an elastic foundation are investigated. Based on the accurate geometrically non-linear theory for Euler–Bernoulli beams, considering the effects of both linear and non-linear elastic foundation, governing equations for large static deformations of the beams subjected to uniform temperature rise are derived. Due to the large deformation of the beam, the constraint forces of elastic foundation in both longitudinal and transverse directions are taken into account. The boundary value problem for the non-linear ordinary differential equations is solved effectively by using the shooting method. Characteristic curves of critical buckling temperature versus elastic foundation stiffness parameter corresponding to the first, the second, and the third buckling mode shapes are plotted. Effects of the elastic foundation stiffness on the post-buckling behaviors are also considered.

2. Mathematical formulations

Consider an elastic beam of initial length l, with uniform cross-sections, resting on a non-linear elastic foundation. The line movements of the two ends are prohibited. A uniform static temperature rise T produces deformation of the beam from its stress free state. By accurately taking into account the axial extension and the curvature of the deformed axial line, we examine the geometrically non-linear response of the beam, and give the non-dimensional governing equations of the problem as follows (Li et al., 2002; Li and Xi, 2006; Li and Zhou, 2003):

$$\frac{\mathrm{d}S}{\mathrm{d}\xi} = \Lambda, \quad \frac{\mathrm{d}U}{\mathrm{d}\xi} = \Lambda\cos\theta - 1, \quad \frac{\mathrm{d}W}{\mathrm{d}\xi} = \Lambda\sin\theta \tag{1a, b, c}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = m, \quad \frac{\mathrm{d}m}{\mathrm{d}\xi} = \Lambda(P_H\sin\theta - P_V\cos\theta) \tag{2a, b}$$

$$\frac{\mathrm{d}P_H}{\mathrm{d}\xi} = \Lambda U(K_1 + K_2(U^2 + W^2)) \tag{3}$$

$$\frac{\mathrm{d}P_V}{\mathrm{d}\xi} = \Lambda W(K_1 + K_2(U^2 + W^2)) \tag{4}$$

$$\Lambda = (P_H\cos\theta + P_V\sin\theta + \tau)/\lambda^2 + 1 \tag{5}$$

The dimensionless quantities in the above equations are defined as follows:

$$(\xi, S, U, W) = (x, s, u, w)/l, \quad \lambda = l(A/I)^{1/2} \tag{6}$$

$$(K_1, K_2) = (k_1, k_2)l^4/EI, \quad \tau = \alpha\lambda^2 T \tag{7}$$

$$(P_H, P_V) = l^2(H, V)/EI, \quad m = lM/EI \tag{8}$$

where x is the central axis of the undeformed beam; $s(x)$ is the length of the deformed central axis with undeformed length x; $u(x)$ and $w(x)$ are the displacements of the central axis in the longitudinal and the transverse directions, respectively; $\theta(x)$ is the angle between the beam axis in deformed state and the x-axis; H and V are the horizontal and vertical internal resultant forces respectively; M is the bending moment; k_1 and k_2 are the linear and cubic stiffness parameters of the elastic foundation, respectively; E is Young's modulus; α is the coefficient of thermal expansion; A and I are the area and the moment of inertia of the cross-section; $\Lambda(x)$ defines the stretching of the initial central axis.

The boundary conditions of a beam with pinned–fixed ends can be written in dimensionless forms as follows:

$$S(0) = 0, \quad U(0) = 0, \quad W(0) = 0, \quad m(0) = 0 \tag{9a}$$

$$U(1) = 0, \quad W(1) = 0, \quad \theta(1) = 0 \tag{9b}$$

In addition to the boundary conditions, a normalization relationship is imposed for the pinned–fixed beam as $\theta(0) = \beta$. Then, for a specified non-vanishing value of β we can determine a thermal post-buckling solution $(S, U, W, \theta, P_H, P_V, m)$ together with the value of the non-dimensional temperature rise τ for a specific buckling mode shape through Eqs. (1)–(4).

3. Numerical method and results

It is difficult to find any analytical solutions to the complicated boundary-value problem (1)–(4) due to the inclusion of strong non-linearity and coupling in it. Therefore, the shooting method is employed to find numerical solutions to the problem. The idea behind the shooting method is to replace the two-point boundary value problem by a sequence of initial value problems. Thus, unknown values of the unknown functions at the initial point are initially estimated to start the computing procedure (William et al., 1986). The Runge–Kutta method is used to integrate the initial problem. At the same time, the Newton–Raphson method is employed to modify the unknowns at the initial point until the boundary conditions at the end point are satisfied.

3.1. Critical buckling and mode transitions

From the physics of the problem, the onset buckling of the beam resting on an elastic foundation is determined by the linearized problem of Eqs. (1)–(4) and can be arrived at by the limit as β tends to zero. Therefore, in determining the values of critical buckling load the effects of the cubic elastic foundation parameter K_2 can be neglected and only Winkler foundation is considered because the displacements are infinitesimal. In the following numerical computation the slenderness is specified as $\lambda = 100$.

Critical buckling temperature rise τ as a function of foundation stiffness parameter K_1 corresponding to different buckling mode shapes for the pinned–fixed beam is plotted in Fig. 1. From it we can find that these

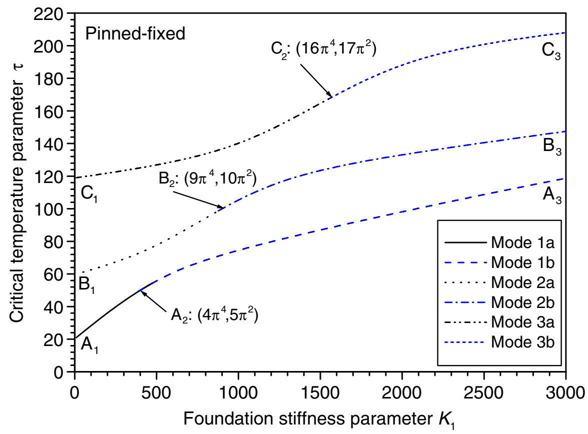

Fig. 1. Characteristic curves of the critical buckling temperature τ, versus foundation stiffness parameter K_1, for the first three buckling modes.

curves have no intersection in the range of $K_1 \leqslant 3000$, which is different from the behaviors of the symmetrically supported beams (Li and Batra, 2005). By analyzing the buckling mode shapes, we find that the three curves consist of six parts, $\overset{\frown}{A_1A_2}$, $\overset{\frown}{A_2A_3}$, $\overset{\frown}{B_1B_2}$, $\overset{\frown}{B_2B_3}$, $\overset{\frown}{C_1C_2}$ and $\overset{\frown}{C_2C_3}$, corresponding to mode 1 (a, b), mode 2 (a, b), and mode 3 (a, b) respectively (see Fig. 2). In the first curve, point A_2 is the first mode transition point

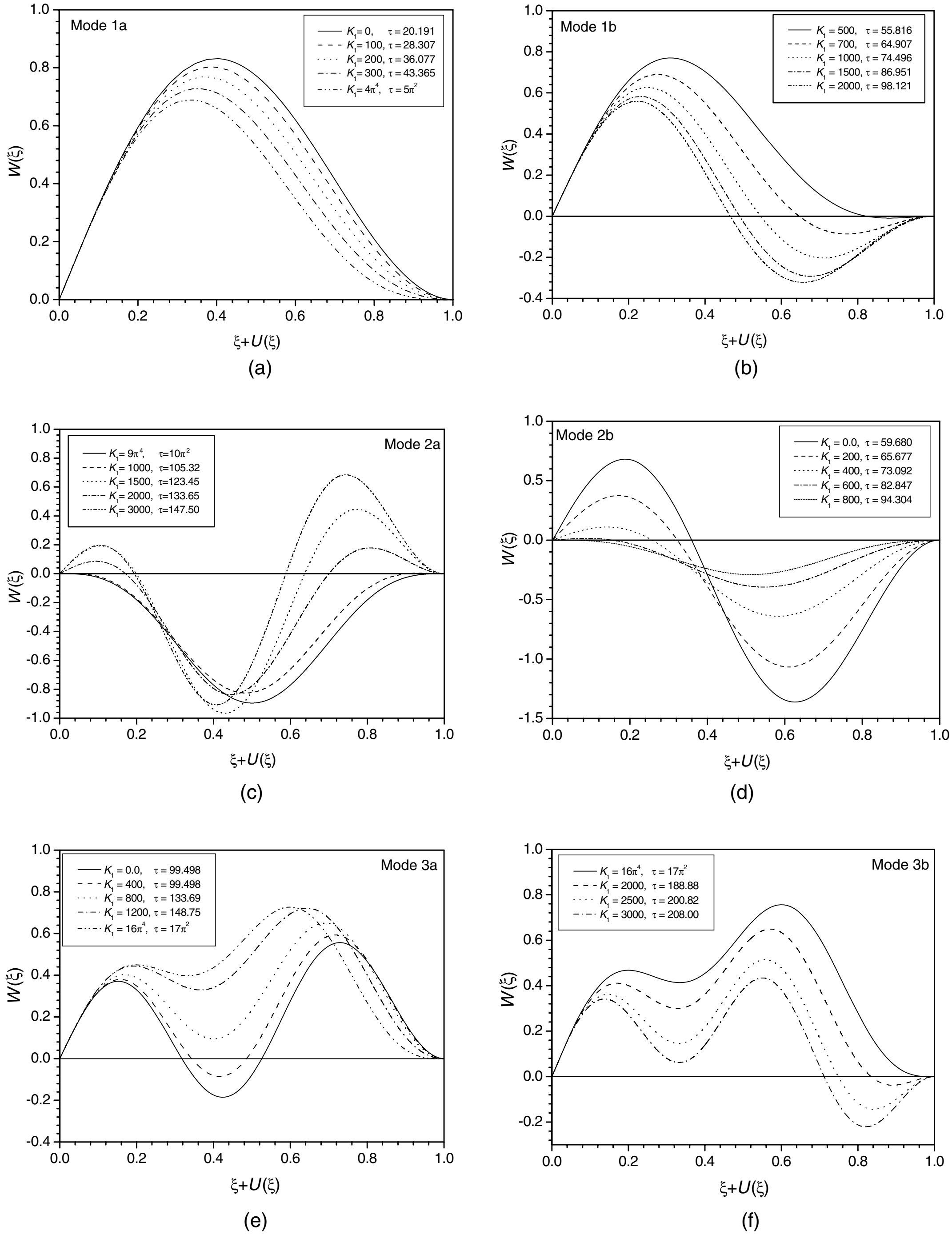

Fig. 2. Critical buckling mode shapes of the pinned–fixed beam for some specified values of K_1. (a) Mode 1a, (b) mode 1b, (c) mode 2a, (d) mode 2b, (e) mode 3a, and (f) mode 3b.

with coordinate $(K_1, \tau)_{T_1} = (4\pi^4, 5\pi^2)$, over which the buckling mode transforms from mode 1a to mode 1b. During this mode transition the value of the end bending moment $m_1 = m(1)$ changes from positive to negative and gets zero at the transition stiffness value $K_1 = (K_1)_{T_1}$. Similar characteristics of mode transition can be found at points B_2 and C_2 with coordinates $(9\pi^4, 10\pi^2)$ and $(16\pi^4, 17\pi^2)$, respectively, from which the second and the third transition stiffness values can be determined as $(K_1)_{T_2} = 9\pi^4$ and $(K_1)_{T_3} = 16\pi^4$, respectively. From Fig. 2 we also see that the value of the end bending moment $m_1 = m(1)$, or the beam curvature $\theta'(1)$, changes from positive to negative and becomes zero at the second and third transition stiffness values.

It needs to explain that the features shown in Fig. 2 are totally different from those of pinned–pinned and fixed–fixed beams with symmetrical supports (Rao and Neetha, 2002; Li and Batra, 2005). From Fig. 2, it can be found that buckling modes of pinned–fixed beam change smoothly and gradually at the transition points. However, the mode transitions of pinned–pinned from mode 1 (symmetrical) into mode 2 (anti-symmetrical), or from mode 2 (anti-symmetrical) into mode 3 (symmetrical) are discontinuous.

3.2. Thermal post-buckling

Thermal post-buckling responses of the beam in the first mode for different values of foundation stiffness can be achieved by using the continuation method by letting parameter β be increased with small steps (Li et al., 2002; Li and Zhou, 2003).

First, effects of the cubic elastic foundation stiffness K_2 on post-buckling temperature and deformation are examined. By giving the linear elastic foundation stiffness $K_1 = 200$, values of post-buckling temperature rise τ changing with the non-linear elastic foundation stiffness parameter K_2, for the different values of end rotational angle β, are listed in Tables 1 and 2 for both $K_2 > 0$ and $K_2 < 0$, respectively, from which we can see that parameter K_2 has a slight influence on the post-buckling temperature τ.

For some given values of K_1, non-dimensional deflection $W(0.4)$, end bending moment $m_1 = m(1)$ and end vertical force $P_V(0)$ varying with temperature rise parameter τ are shown in Figs. 3–5. These curves can be considered as the equilibrium paths of the heated beam corresponding to the first post-buckling mode (mode 1). It can be found that the end force increases rapidly with the increase of foundation stiffness.

In Fig. 6, post-buckling equilibrium configurations of the beam in the first mode corresponding to different pairs of values of (K_1, τ) are shown for a given value of end rotational angle $\beta = 10^{\circ}$, from which we can see

Table 1
Post-buckling temperature τ changes along with the end rotational angle, $\theta(0)$ of the pinned end and the non-linear elastic foundation stiffness parameter K_2 ($K_1 = 200, K_2 \geqslant 0$)

$\theta(0)$ (°)	K_2					
	0	500	1000	3000	5000	10000
1	36.534	36.534	36.535	36.537	36.538	36.543
5	46.909	46.919	46.929	46.969	47.010	47.112
10	79.390	79.417	79.444	79.553	79.661	79.928
15	133.73	133.74	133.75	133.79	133.82	133.92
20	210.23	210.11	210.00	209.54	209.10	208.05

Table 2
Post-buckling temperature τ changes along with the end rotational angle, $\theta(0)$ of the pinned end and the non-linear elastic foundation stiffness parameter K_2 ($K_1 = 200, K_2 \leqslant 0$)

$\theta(0)$ (°)	K_2					
	0	−500	−1000	−3000	−5000	−10000
1	36.534	36.533	36.533	36.531	36.529	36.525
5	46.909	46.898	46.888	46.847	46.806	46.704
10	79.390	79.363	79.335	79.225	79.114	79.835
15	133.73	133.72	133.71	133.67	133.63	133.52
20	210.23	210.34	210.46	210.94	211.42	212.68

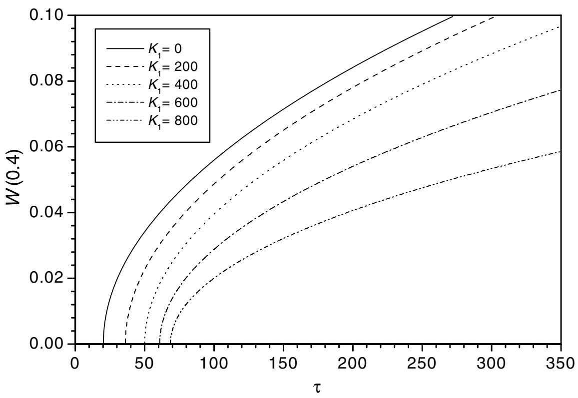

Fig. 3. Dimensionless deflection, $W(0.4)$, versus temperature rise τ for some specified values of K_1 ($K_2 = 0$, mode 1).

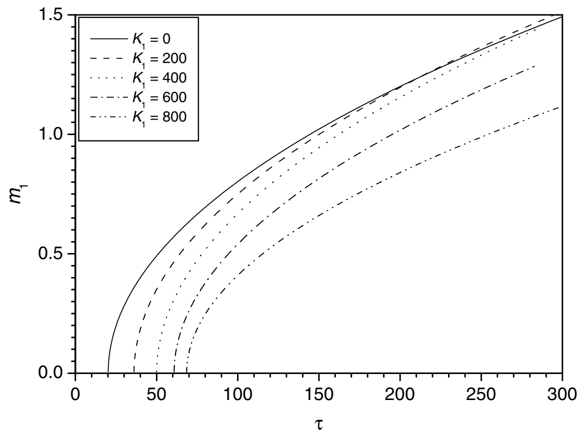

Fig. 4. End bending moment m_1 versus temperature rise τ for some values of K_1 ($K_2 = 0$, mode 1).

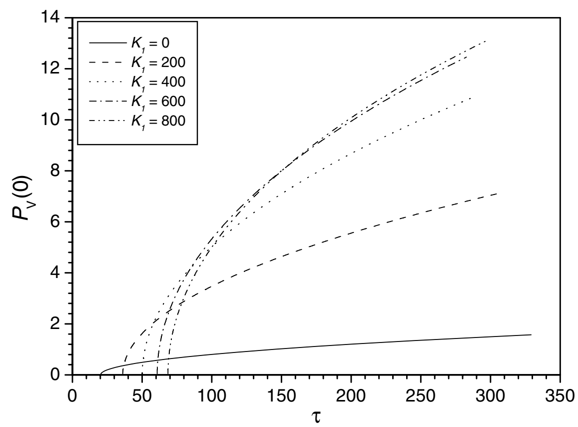

Fig. 5. $P_V(0)$ versus temperature rise τ for different values of K_1 ($K_2 = 0$, mode 1).

that the post-buckling mode shapes change with the increase of the foundation stiffness parameter K_2. Furthermore, in Fig. 7, we plotted the post-buckling configurations corresponding to different values of β and

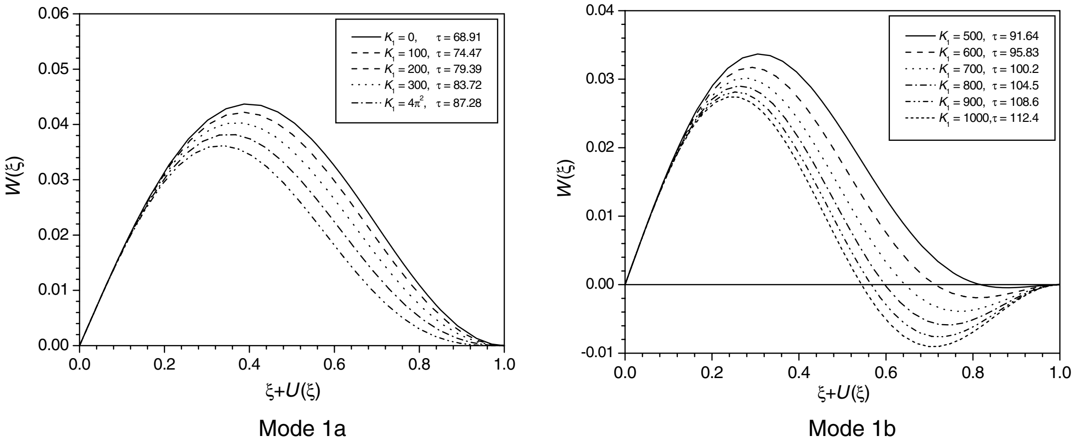

Fig. 6. Thermal post-buckling configurations of the beam in the first mode for different values of K_1 and τ ($K_2 = 0, \beta = 10°$).

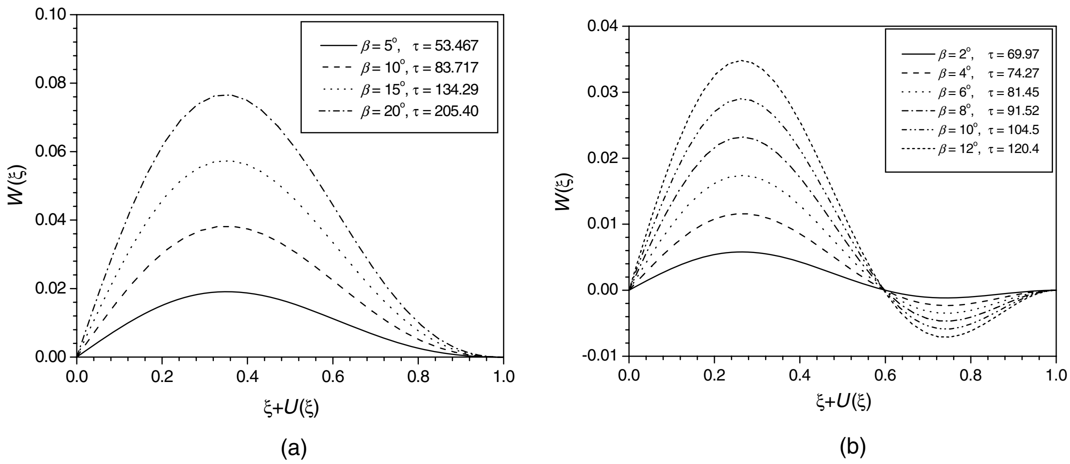

Fig. 7. Thermal post-buckling configurations of the beam in the first mode for different values of β and τ. (a) $K_1 = 300, K_2 = 0$ and (b) $K_1 = 800, K_2 = 0$.

τ for $K_1 = 300$, 800. This figure shows that for a given value of K_1 the beam deformation increases with the rise of temperature. However, the post-buckling mode shapes do not change with the increase of τ. Especially, we can see that the zero deflection point in Fig. 7(b) remains the same in the course of the development of the deformation.

4. Conclusions

Both thermal critical buckling and post-buckling of beams with pinned–fixed ends and resting on a non-linear elastic foundation are presented. The constraint forces of elastic foundation in both longitudinal and transverse directions are taken into account. Boundary value problem for the non-linear ordinary differential equations are solved effectively by using the shooting method. Characteristic curves of critical buckling temperature versus the foundation stiffness parameter are plotted corresponding to the first, the second, and the third shape modes, from which the first, the second, and the third transition foundation stiffness for the beam on Winkler foundation are numerically evaluated. It is as we expect that the non-linear foundation stiffness parameter has no sharp influence on the critical buckling temperature and it has a slight effect on the post-buckling temperature compared with the linear foundation parameter. The linear elastic foundation stiffness parameter has obvious influence on both the critical buckling and post-buckling configuration modes. Nevertheless, if the linear foundation stiffness is fixed, the post-buckling mode shapes do not change but the level of the deformation develops with the temperature rise.

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